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An Update on the Isospin-Breaking Effects in the Pion Decay Constant with Staggered Quarks

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Alessandro Cotellucci[a,*] **and Davide Giusti**[a,b] **on behalf of the Budapest-Marseille-Wuppertal collaboration**

[a] *Jülich Supercomputing Centre, Forschungszentrum Jülich, D-52428 Jülich, Germany*
[b] *Fakultät für Physik, Universität Regensburg, 93040, Regensburg, Germany*

E-mail: a.cotellucci@fz-juelich.de

We present an update on the ongoing computation of the isospin-breaking effects in the Pion Decay Constant from the BMW Collaboration. The calculation is carried out with N_f=2+1+1 staggered quarks with a near-physical pion mass and $\mathrm{QED_L}$. We give an update on the isosimmetric value and the current determination used to compute the gradient-flow scale w_0, then we present some preliminary results of the valence-valence contribution to the axial-pseudoscalar correlator for different volumes and lattice spacings. We also discuss the next steps and plans.

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[*] Speaker

1. Motivations

The unitarity of the first row of the CKM matrix is an open problem in precision flavour physics, as shown in Figure 1 from [1]. At a current level of precision, isospin-breaking effects become relevant.

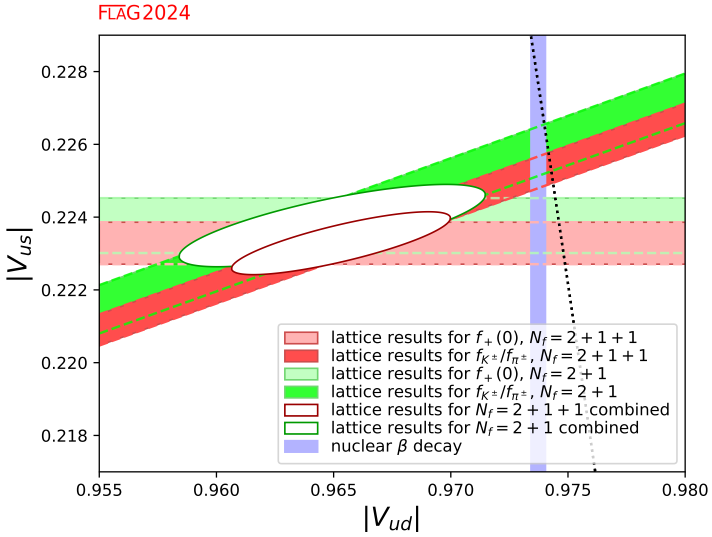

Figure 1: Status of the determination of the V_{ud} and V_{us} elements of the CKM matrix as reported by the Flavour Lattice Averaging Group [1].

In Isosymmetric QCD the pion decay constant f_π is defined by the hadronic matrix element:

$$\langle 0|\,\bar{u}\gamma_4\gamma_5 d\,|\pi(\vec{0})\rangle = M_\pi f_\pi . \tag{1}$$

To compute the pion decay constant in QCD+QED F_π one needs to consider the leptonic decay rate $\pi^+ \to l^+\nu_l(\gamma)$, in the PDG parametrization it is:

$$F_\pi^2 = \frac{\Gamma_{\pi^+\to l^+\nu_l(\gamma)}}{\frac{G_F^2}{8\pi}|V_{ud}|^2 M_\pi m_l^2\left(1-\frac{m_l^2}{M_\pi^2}\right)^2} = f_\pi^2\left[1+\delta R_\pi\right] \tag{2}$$

with δR_π the isospin-breaking correction to the decay constant. The full computation of δR_π requires the renormalization of the electroweak hamiltonian:

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V_{ud}^* \left[\bar{d}\gamma^\mu\left(1-\gamma_5\right)u\right]\left[\bar{\nu}_l\gamma_\mu\left(1-\gamma_5\right)l\right] . \tag{3}$$

There is one lattice computation of the isospin-breaking corrections in the pion and kaon decay constants by the RM123-Southampton collaboration [2]. In the ratio f_K/f_π the renormalization constant cancel (in a massless scheme) so there are more lattice computations of the correction to the latter, from the RM123-Southampton collaboration [3], from RBC-UKQCD [4] and a more recent from RBC-UKQCD with QED$_\infty$ [5].

The pion decay constant F_π is used for the scale setting in the BMW/DMZ determination of the anomalous magnetic moment of the muon [6].

2. Mixed Determination

The isospin-breaking effects in the pion decay constant are computed by combining the work from the RM123-Southampton collaboration [2] and the BMW results for isosymmetric QCD and the sea quark IBE.

The dependency of the pion decay constant in units of w_0 on the quark electric charge and bare quark masses is parametrized as follows (up to higher order effects):

$$\begin{aligned} w_0F_\pi =& A + BF_\pi^{-2}M_{\pi^+}^2 + CF_\pi^{-2}\left(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2\right)/2 \\ &+ Ee_v^2 + Fe_ve_s + Ge_s^2 \end{aligned} \tag{4}$$

where e_v and e_s are, respectively, the valence and sea quark charges. The coefficients B and C are mass derivatives and are computed in QCD. The coefficients E, F and G are the valence-valence, sea-valence and sea-sea derivatives.

Once all the coefficients in Eq. (4) are determined, the parametrization can be split into the QCD+seaQED and the valQED part:

$$[w_0F_\pi]_{\text{QCD+QED}} = [w_0F_\pi]_{\text{QCD+seaQED}} + [w_0F_\pi]_{\text{valQED}}\,, \tag{5}$$

both of the terms on the right-hand side are dependent on the scheme used to define QCD. It is fundamental to remember that w_0 has not valence quak isopsin-breaking effects so $[w_0F_\pi]_{\text{valQED}}$ includes only the valence quark IBE on F_π. The quantity $[w_0F_\pi]_{\text{QCD+seaQED}}$ is determined with the BMW data, but the valence contribution is computed from the RM123-Southampton result[1].

To combine the two results, the same scheme has to be employed. The work [2] is defined in the GRS scheme by the quantities [7]:

$$\begin{aligned} [M_\pi]^{\text{GRS}} =& 135.0(2)\text{ MeV} \\ \left[\frac{1}{2}\left(M_{K^\pm} + M_{K^0}\right)\right]^{\text{GRS}} =& 494.6(1)\text{ MeV} \\ [f_\pi]^{\text{GRS}} =& 130.65(12)\text{ MeV} \\ [\delta R_\pi]^{\text{GRS}} =& 0.0153(19)^{2}. \end{aligned} \tag{6}$$

Since they were working in the electroquenched approximation, which neglects the sea quark IBE, the QCD result includes the sea quark isospin-breaking effects following:

$$[*]^{\text{GRS}} = [*]^{\text{phys}}_{\text{QCD+QED}} - [*]^{\text{GRS}}_{\text{valQED}} = [*]^{\text{GRS}}_{\text{QCD+seaQED}}\,. \tag{7}$$

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[1] The valence QED contribution to the pion decay rate from Ref. [2] is given in a GRS scheme in which the quark masses, renormalized at $\mu = 2$ GeV in $\overline{\text{MS}}$, remain constant as the electromagnetic coupling is turned on or off and the overall lattice scale is fixed using f_π. In the electroquenched approximation adopted there valence QED corrections do not affect the renormalization of the strong coupling – *i.e.* the GRS condition $[\widehat{g_s}]_{\text{QCD}}(\mu) = [\widehat{g_s}]_{\text{QCD+valQED}}(\mu)$ $(*)$, where $\widehat{\ }$ indicates the quantities renormalized in the respective theories, holds trivially – and thus do not change the lattice spacing from its pure QCD value at fixed β. We stress that in the full QCD+QED calculation presented here the lattice scale is consistently set using F_π and, since the purely gluonic quantity w_0 is free of valence QED contributions – *viz.* a relation similar to $(*)$ occurs for w_0 as well –, it is justified to use the value of δR_π from Ref. [2] as input in our computation to evaluate the term $[w_0F_\pi]_{\text{QCD+valQED}}/[w_0F_\pi]_{\text{QCD}}$ and Eq. (8) without introducing scheme ambiguities.

[2] The value used for our determination of w_0F_π is $\delta R_\pi = 0.0150(18)$, it does not include the original χPT estimate of the sea-quark IBE.

Using the parametrization (4), the identity (7) can be used to evaluate $[w_0F_\pi]^{\text{GRS}}_{\text{QCD+seaQED}}$ which can then be combined with the RM123S results using the relation, up to negligible $O\left(\alpha_{\text{em}}^2\right)$ corrections:

$$[w_0F_\pi]_{\text{QCD+QED}} = [w_0F_\pi]^{\text{GRS}}_{\text{QCD+seaQED}}\sqrt{1+[\delta R_\pi]^{\text{GRS}}}. \tag{8}$$

3. Isosymmetric QCD value in the FLAG scheme

The isosymmetric QCD value of $w_0 f_\pi$ is computed using the ensembles of the BMW collaboration shown in Figure 2 with QCD tree-level Symanzik action for the gauge field and N$_f$=2+1+1 Staggered fermions coupled with 4 levels of stout smearing with radius $\rho = 0.125$. The decay constant is computed from the pseudoscalar two-point function through the chiral WTI.

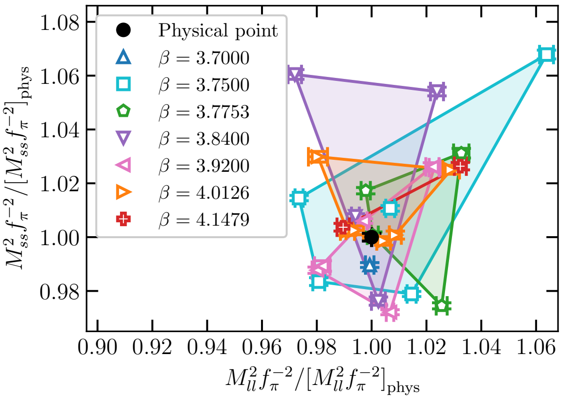


Figure 2: Landscape of the ensembles used for the isoQCD determination of $w_0 f_\pi$ around the physical point from [6].

The analysis follows the strategy presented in [8]. To have a better control over the continuum extrapolation we add the determination of w_0 using the Zeuthen-flow [9] in addition to the Wilson-flow [10], Figure 3 shows the continum extrapolation of $w_0 f_\pi$ in the lattice spacing and in the taste-breaking term and the relative probability distribution function of both the statistical and sistematic effect. The result is presented in the FLAG scheme [11].

Table 1 shows the breakdown of the total error on $w_0 f_\pi$. The uncertainty is dominated by the systematic error and, in particular, the biggest contribution is represented by the continuum extrapolation under the entries: *Type of gradient flow*, *Lattice spacing cuts* and *Order of fit polynomials*. To improve the determination, we are generating data at a finer value of the lattice spacing. Other sources of systematic error listed are: *Pseudoscalar fits* (effects of the fit range of the effective pion decay constant) and *Finite volume χPT order* (the difference in the finite volume effects between NLO and NNLO χPT).

The isosymmetric part of $w_0 f_\pi$ is determined with a 0.23% precision, thus making it necessary to include IBE.

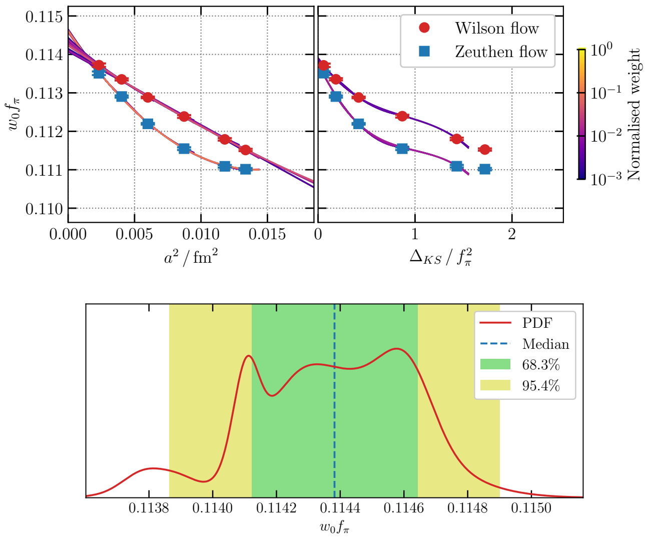


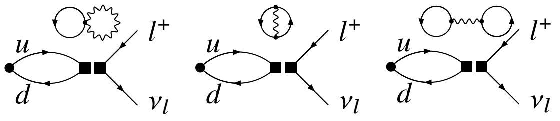
Figure 3: Continuum limit extrapolation of the iso QCD value of $w_0 f_\pi$. The upper plot represents the continuum extrapolation both in the lattice spacing and in the taste-symmetry breaking terms Δ_{KS} (computed by comparing the masses of different meson tastes on our ensembles); in the lower plot the probability distribution function (PDF) including both statistical and systematic variations is shown in red.

Median	0.11438	
Total error	0.00026	0.22736 %
Statistical error	0.00012	0.10786 %
Systematic error	0.00023	0.20015 %
Type of gradient flow	0.00013	0.10966 %
Pseudoscalar fits	0.00004	0.03144 %
Finite volume χPT order	0.00001	0.01023 %
Lattice spacing cuts	0.00006	0.04956 %
Order of fit polynomials	0.00019	0.16215 %

Table 1: Error budget of the isoQCD value of $w_0 f_\pi$. The first row contains the median value, from the second on the errors are presented with the value and the percentage relative value. The last five rows present a breakdown of the systematic error on $w_0 f_\pi$.

4. Sea Quark Isospin-Breaking Effects

As mentioned in Section 2, the sea quark isospin-breaking effects to $w_0 F_\pi$ are determined using the BMW ensemble. The sea-sea diagrams contributing to the electromagnetic derivative of the pion decay constant are:



. (9)

The renormalized sea quark electromagnetic derivative is computed by cancelling the UV-divergence with the one coming from the scheme observables in the parametrization (4) following the convention from [12].

The diagrams are estimated using QED_L in Coulomb gauge and the ensembles listed in Table 2.

action	β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#meas
`4stout`	3.7000	0.1315	24×48	volume/24	0.057291	27.899	716
			48×64	volume/48	0.057291	27.899	300
	3.7753	0.1116	28×56	dir00	0.047615	27.843	887
	3.8400	0.0952	32×64	dir00	0.043194	28.500	1110
				dir02	0.043194	30.205	1072
				dir04	0.040750	28.007	1036
				dir05	0.039130	26.893	1035
`4hex`	0.7300	0.1120	56×84	phys2/56	0.06061	33.728	1305

Table 2: Set of ensembles used for the computation of the sea quark isospin-breaking effects.

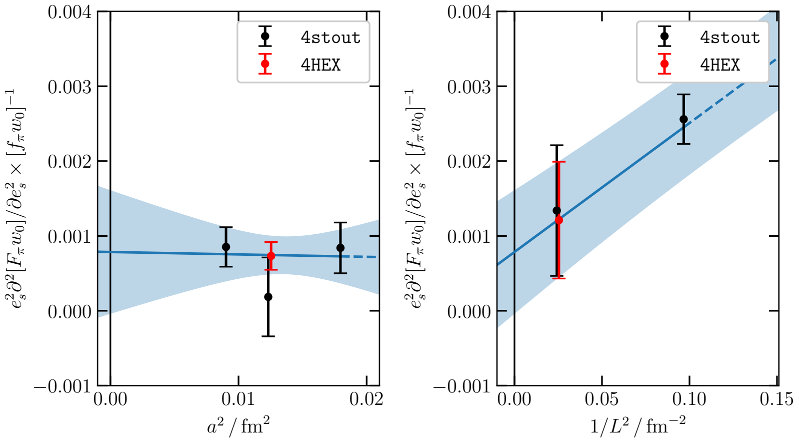
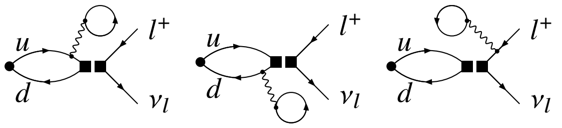


Figure 4: Renormalized sea quark electromagnetic derivative of $w_0 F_\pi$ divided by its QCD value. The left panel shows the continuum, the right panel the infinite volume extrapolation.

Figure 4 shows the effect of the sea derivative relative to the isosymmetric QCD value of $w_0 F_\pi$. As can be seen by the plot, the sea quark IBE in the continuum is about 0.1% of the isosymmetric value, and it shows a flat behaviour in the continuum limit and power-law finite volume effects of order $O\left(1/L^2\right)$ as the leading order structure-dependent finite volume effects [13].

The sea-valence IBE, including the following diagrams:



, (10)

are estimated to be of the order of 20% compared to the sea-sea derivative and thus neglected in our computation.

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5. Total Error Budget

Combining the results for the isoQCD and the IBE as mentioned in Section 2 we obtain the following preliminary QCD+QED value of $w_0 F_\pi$:

$$[w_0 F_\pi]_{\text{QCD+QED}} = 0.11527(15)(26)[31] \text{ (Preliminary)}, \tag{11}$$

where the first error is the statistical, the second is systematic, and the latter is the combined total error. The systematic error consists of the following elements:

$$(26) = (23)_{\text{QCD}}(11)_{\text{QED-Valence}}(5)_{\text{QED-Sea}} \text{ (Preliminary)}. \tag{12}$$

The QCD error dominates the final uncertainty, while the QED error is dominated by the valence QED value.

Combining $w_0 F_\pi$ with the QCD+QED value of $F_\pi = 131.711(45)$ MeV (obtained using $V_{ud} = 0.97367(32)$ and $\Gamma = 3.8408(7) \times 10^7\ \text{s}^{-1}$ from [14]) we get the following value of w_0:

$$[w_0]_{\text{QCD+QED}} = 0.17270(22)(40)[46] \text{ fm (Preliminary)}. \tag{13}$$

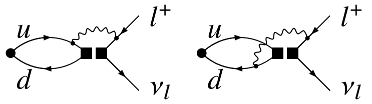
The systematic error is made by the following contributions:

$$(40) = (34)_{\text{QCD}}(17)_{\text{QED-Valence}}(8)_{\text{QED-Sea}}(7)_{\text{exp}} \text{ (Preliminary)}, \tag{14}$$

where the additional systematic error *exp* accounts for the experimental value of F_π. We are working on improving the QCD error and improving the valence QED determination by providing an independent determination of δR_π and by checking the power-law finite volume effects in light of the work from [4].

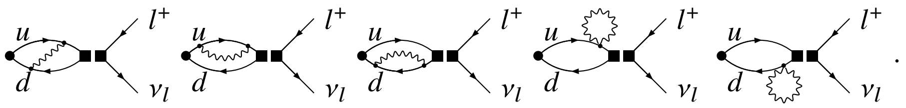
6. Ongoing Work on the Valence Quark Isospin-Breaking Effects

The valence-quark IBE in the pion decay constant requires the computation of the non-factorizable diagrams:



(15)

(actively involving the lepton) and the factorizable ones; we present some preliminary results of the factorizable diagrams. The factorizable valence contributions to the electromagnetic derivative of the pion decay constant are given by the following diagrams:



(16)

The diagrams are computed using sequential propagators and 16 photon sources in Coulomb gauge and 3 U(1) random sources. The zeroth mode of the photon is removed following the QED_L prescription. We use the ensembles listed in Table 3. We tested the program by comparing it with numerical derivatives at a fixed random source and photon source.

The correction is obtained by expanding the axial-pseudoscalar (using the conserved axial current) and pseudoscalar-pseudoscalar two-point function:

$$\frac{G_{ud}^{PA_4}(x_4)^2}{G_{ud}^{PP}(x_4)} = M_\pi Z_A^{-2} F_\pi^2 e^{-M_\pi T/2} \frac{\sinh^2\left[M_\pi\left(T/2 - x_4 - 1/2\right)\right]}{\cosh\left[M_\pi\left(T/2 - x_4\right)\right]} \tag{17}$$

where T is the temporal exension and Z_A is the renormalization constant which is not considered in these results and, thus, it is left for future work.

action	β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
`4stout`	3.7000	0.1315	24×48	volume/24	0.057291	27.899	48
			32×64	volume/32	0.057291	27.899	48
			48×64	volume/48	0.057291	27.899	48
			64×64	volume/64	0.057291	27.899	48
	3.8400	0.0952	64×96	dir00	0.043194	28.500	48

Table 3: Set of ensembles used for the computation of the valence quarks isospin-breaking effects.

Figure 5 shows the preliminary result for the effective factorizable valence derivative for the four different volumes considered.

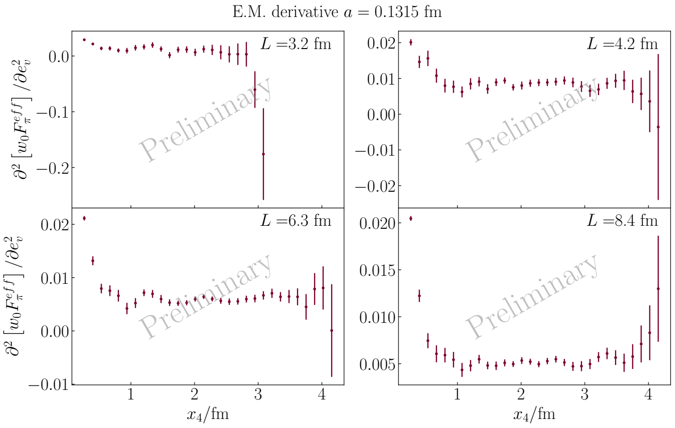


Figure 5: Factorizable valence electromagnetic derivative of $w_0 F_\pi$ for $a = 0.1315$ fm. The preliminary results for the four volumes are shown.

Figure 6 shows the preliminary result for the effective factorizable valence derivative for the finer value of the lattice spacing. The effective derivative presents a clear plateau at large time separation, showing little excited states contamination and little oscillating contribution.

7. Conclusions & Outlook

In this Proceedings, we have presented the results of the current determination of the pion decay constant in QCD+QED by the BMW collaboration for the determination of the scale setting quantity w_0, the preliminary value is:

$$[w_0]_{\text{QCD+QED}} = 0.17270(22)(40)[46] \text{ fm (Preliminary)}, \tag{18}$$

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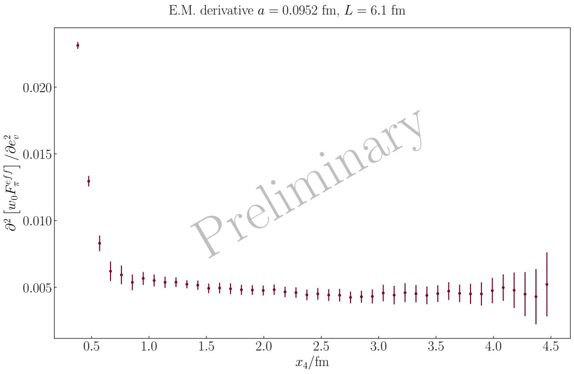

Figure 6: Factorizable valence electromagnetic derivative of $w_0 F_\pi$ for $a = 0.0952$ fm.

where the precision is dominated by the continuum extrapolation of the isosymmetric part. The determination of isospin-breaking effects was done by combining sea quark effects from BMW ensembles and the valence-quark effects from the RM123 work.

We are working on improving the continuum extrapolation (by adding a finer lattice spacing) and on a completely independent determination of the valence effects, which will include the sea-valence (that have been estimated to be suppressed). This requires the determination of the non-factorizable diagrams involving the leptons and the renormalization of the electroweak Hamiltonian. The authors of [4, 15] showed the poor convergence of the series of finite volume effects to the pion decay constant in $\mathrm{QED_L}$, for this reason, we plan a detailed study of the finite volume effects using different volumes (up to 10.8 fm).

Acknowledgments

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