PROCEEDINGS OF SCIENCE

On the status of the 4-loop splitting functions in QCD

S. Moch[a,*] **and A. Vogt**[a,b,*]

[a] *II. Institute for Theoretical Physics, Hamburg University, Luruper Chaussee 149, D-22761 Hamburg, Germany*

[b] *Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, United Kingdom*

E-mail: sven-olaf.moch@desy.de, Andreas.Vogt@liverpool.ac.uk

We briefly discuss recent results on the four-loop splitting functions for the scale dependence of the parton distributions of hadrons. The three non-singlet cases are now completely known. The results agree with almost all expectations, but exhibit some small-x double logarithms not encountered at lower orders. The flavour-singlet cases include, as already obvious from the present partial results, analytical structures not seen in the splitting functions up to now. The numerical effects (and their uncertainties, in the singlet cases) of the four-loop contributions to the scale derivatives amount to about 1% or less down to momentum fractions x as low as about 10^{-4} at a standard benchmark point with $\alpha_s = 0.2$, with the exception of the total valence quark distribution.

Loops and Legs in Quantum Field Theory (LL2026)
12-17, April, 2026, Bayreuth, Germany

*Speaker

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0) All rights for text and data mining, AI training, and similar technologies for commercial purposes, are reserved. ISSN 1824-8039 . Published by SISSA Medialab.

PoS(LL2026)022

1. Introduction

Quark and gluon parton distribution functions (PDFs) are indispensable ingredients for all analyses of hard-scattering observables in collisions involving initial-state hadrons. The PDFs are not calculable in perturbative QCD, unlike their scale dependence (evolution)

$$\frac{\partial}{\partial \ln \mu^2} f_i(x,\mu^2) \;=\; \sum_k \left[P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](x) \;. \tag{1.1}$$

Here $\otimes$ abbreviates the Mellin convolution in the momentum fraction x, and μ is the factorization and renormalization scale, usually defined in the $\overline{\rm MS}$ scheme. This set of $(2n_f+1) \times (2n_f+1)$ integro-differential equations, where n_f is the number of light quark flavours, can be decomposed into $(2n_f-1)$ scalar (non-singlet) equations for combinations of (anti-) quark distributions decoupled from each other and the gluon distribution $g(x,\mu^2)$ and the 2×2 singlet-quark gluon (q_s, g) system.

The corresponding splitting functions are

$$P_{\rm ns}^{\pm}\,, \quad P_{\rm v} \;=\; P_{\rm ns}^{-} + P_{\rm ns}^{\rm s} \quad \text{and} \quad \mathbf{P} \;=\; \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \;\text{ with }\; P_{\rm qq} \;=\; P_{\rm ns}^{+} + P_{\rm ps} \tag{1.2}$$

for flavour differences of quark- antiquarks sums (+) and differences (-), the total valence distribution (v) and the (q_s, g) system. Their expansion in powers of the strong coupling α_s can be written as

$$P_a(x,\alpha_s) \;=\; \sum_{m=0} P_a^{(m)}(x)\, a_s^{m+1} \quad \text{with} \quad a_s \;=\; \alpha_s(\mu^2)/(4\pi) \;. \tag{1.3}$$

The even- or odd-N moments of the splitting functions are related to the anomalous dimensions $\gamma_a^{(n)}(N)$ of twist-two operators in the light-cone expansion,

$$\gamma_a^{(n)}(N) \;=\; -P_a^{(n)}(N) \;=\; -\int_0^1 dx\, x^{N-1} P_a^{(n)}(x) \;, \tag{1.4}$$

where the relative sign is a standard convention. In general, eq. (1.4) holds at even N for $a = +$ and the singlet (s) cases and at odd N for $a = -, {\rm v}$. The remaining moments of the splitting functions are not accessible via the operator-product expansion (OPE) or related approaches; they involve higher values of the Riemann ζ-function than the above 'natural' moments.

Together with the $n^{\rm th}$-order corrections to the lowest-order partonic cross sections, the sum to $m = n$ in eq. (1.3) form the (next-to-)^n-leading order (N^{n}LO) approximation of perturbative QCD. The 3-loop ($n = 2$) splitting functions, required for fully consistent N^{2}LO analyses which are now the standard for many processes, were computed more than 20 years ago [1, 2].

N^{3}LO calculations are desirable or required for very high accuracy, such as the determination of PDFs and the strong coupling α_s from deep-inelastic scattering (DIS) [3–6], or for processes with large higher-order corrections, such as Higgs-boson production in proton-proton collisions [7–9]. First corresponding 4-loop contributions to all splitting functions have been presented at a previous Loops & Legs conference, 10 years ago [10]. Since then a considerable amount of research has been devoted to these functions [11–31], for now culminating in the completion of all three non-singlet splitting functions $P_{\rm ns}^{(3)a}(x)$, $a = \pm, s$, in eqs. (1.2) and (1.3) in ref. [32]. In this contribution, we briefly address a few aspects of these non-singlet results, for a more detailed discussion see ref. [33], as well as the present status in the flavour-singlet sector.

PoS(LL2026)022

2. The non-singlet cases

2.1 General properties

Until a few months ago, only the fermionic (n_f) contributions to $P_{\rm ns}^{(3)\pm}(x)$ were fully known in QCD and its gauge-group generalizations [11, 29]. Before ref. [32], the exact expressions for the main (n_f^0) contribution had been known only in the limit of a large number of colours n_c [12].

The corresponding anomalous dimensions to now (at least) four loops can be cast in the form

$$\gamma_{\rm ns}^{(\ell)\pm}(N) \;=\; \sum_{n=0}^{2\ell-1} \sum_{a=0,1} \sum_{k=0}^{2\ell+1-n} \sum_{\vec{w},\, w=0}^{2\ell+1-n-k} c_{akw}^{(\ell,n)}\; \zeta_n \; \frac{1}{(N+a)^k}\; S_{\vec{w}}(N) \tag{2.1}$$

in terms of harmonic sums recursively defined by [34]

$$S_{\pm m}(N) \;=\; \sum_{n=1}^{N} (\pm 1)^n \, \frac{1}{n^m}\;, \quad S_{\pm m_1,\, m_2,\, \ldots,\, m_d}(N) \;=\; \sum_{n=1}^{N} (\pm 1)^n \, \frac{1}{n^{m_1}}\, S_{m_2,\, \ldots,\, m_d}(n)\;. \tag{2.2}$$

The weight w of a sum is given by the sum of the absolute values of its indices m_i. The sum over $\vec{w}$ in eq. (2.1) excludes any sums with an index -1 (see below). ζ_n are values of Riemann's ζ-function; the non-ζ terms are included in eq. (2.1) via $\zeta_0 \equiv 1$. The sum over n excludes $n = 1, 2$. The case of $\gamma_{\rm ns}^{\rm s}(N)$ is analogous, but the sum over the denominator shifts a also includes $a = -1$ and $a = 2$.

The conformal symmetry of QCD for $D = 4 - 2\varepsilon$ dimensions at a particular value of the coupling constant implies the relation [35–37], see also ref. [38],

$$\gamma_{\rm ns}^{a}(N) \;=\; \gamma_{\rm u}^{a}\left(N + \sigma\, \gamma_{\rm ns}^{a}(N) - \beta(a_{\rm s})/a_{\rm s}\right)\;, \tag{2.3}$$

with $\beta(a_{\rm s}) = -\beta_0\, a_{\rm s}^2 - \beta_1\, a_{\rm s}^3 - \ldots$, and $\sigma = -1$ for the present initial-state (space-like) splitting functions and $\sigma = 1$ for their final-state (time-like) fragmentation counterparts. Eq. (2.3) leads to

$$\begin{aligned}
\gamma_{\rm u} \;=\; \sum_{n=0} a_{\rm s}^{n+1}\, \gamma_{\rm u}^{(n)} \;=\;& a_{\rm s}\, \gamma_0 \\
&+ a_{\rm s}^2 \left(\overline{\gamma}_1 \;-\; \beta_0\, d_N \gamma_0\right) \\
&+ a_{\rm s}^3 \left(\overline{\gamma}_2 - \frac{1}{6}\, d_N^2 \gamma_0^3 - \beta_1\, d_N \gamma_0 - \beta_0\, d_N \overline{\gamma}_1 + \frac{1}{2}\, \beta_0^2\, d_N^2 \gamma_0\right) \\
&+ a_{\rm s}^4 \left(\overline{\gamma}_3 - \frac{1}{2}\, d_N^2 (\gamma_0^2\, \overline{\gamma}_1) - \beta_2\, d_N \gamma_0 - \beta_1\, d_N \overline{\gamma}_1 - \beta_0\, d_N \overline{\gamma}_2 \right. \\
&\qquad \left. + \frac{1}{6}\, \beta_0\, d_N^3 \gamma_0^3 + \beta_0 \beta_1\, d_N^2 \gamma_0 + \frac{1}{2}\, \beta_0^2\, d_N^2 \overline{\gamma}_1 - \frac{1}{6}\, \beta_0^3\, d_N^3 \gamma_0\right) \\
&+ O(a_{\rm s}^5)\;,
\end{aligned} \tag{2.4}$$

where all arguments N have been suppressed for brevity. $\overline{\gamma}_n$ denotes the average of the $\sigma = -1$ and $\sigma = 1$ expansion coefficients. Since the difference between the time-like and space-like non-singlet splitting functions is fixed by lower-order information, see refs. [12, 39] and references therein, eq. (2.4) can be written purely in terms of the present $\sigma = -1$ parton-evolution splitting functions.

We have evaluated the derivatives $d_N^m \equiv d^m/dN^m$ in eq. (2.4) via inverse Mellin transforms to x-space, where they correspond to products with $\ln^m x$, and subsequent Mellin transforms of these products in Form [40–42] using packages first provided with ref. [40].

PoS(LL2026)022

A salient point of eq. (2.4) is that the universal evolution kernels $\gamma_{\rm u}^{(n)a}(N)$ are reciprocity respecting (RR), i.e., their Mellin inverses fulfil $P_{\rm u}^{(n)a}(x) = -xP_{\rm u}^{(n)a}(1/x)$. For the 'unsummed' denominators analogous to eq. (2.1) this implies that they can occur only in the combinations

$$\eta \;=\; \frac{1}{N} - \frac{1}{N+1}\;, \quad \nu \;=\; \frac{1}{N-1} - \frac{1}{N+2}\;, \tag{2.5}$$

respectively corresponding to $1-x$ and $x^{-1}-x^2$, which are invariant under $N \to -N-1$. The harmonic sums can enter only in suitable linear combinations which can be expressed in terms of

$$\mathbb{S}_{m_1,\ldots,m_d}(N) \;=\; (-1)^N \sum_{k=1}^{N} (-1)^k \binom{N}{k}\binom{N+k}{k} S_{m_1,\ldots,m_d}(k)\;, \tag{2.6}$$

where all indices are positive. These binomial sums, already considered at $w \le 4$ in ref. [34], provide a basis of all RR linear combinations of the harmonic sums (2.2).

Other bases are possible which exploit the fact that products of RR sums are again RR. Since they do not lead to (significantly) shorter expressions at high weights, it seems easiest to use eq. (2.6). A set of equations for converting RR functions given in terms of standard harmonic sums up to weight 7 to binomial sums can be found in app. B of ref. [33].

The fact the $\gamma_{\rm u}^{(n)a}(N)$ in eq. (2.4) can be expressed in terms of the quantities (2.5) and (2.6) has direct consequences on the large-N ($x \to 1$) limit. This has been discussed in refs. [32, 33], we do not need to repeat it here. It may be worthwhile to note, however, that this RR structure provides a partial explanation for the observation that sums with index -1 do not occur in any splitting functions in massless perturbative QCD (we do not recall any derivation of this in the literature): there are no RR combinations of harmonic sums that involve any sums with index -1. Hence such sums can enter neither $\gamma_{\rm u}^{(n)a}$ nor, after inverting eq. (2.4), the non-singlet anomalous dimensions in the $\overline{\rm MS}$ scheme. The analogous statement in the singlet sector is, at this point, restricted to the trace and determinant of the matrix in eq. (1.2), see ref. [31].

2.2 Small-x expansions

The most intriguing aspect of the new results for $\gamma_{\rm ns}^{(n)a}$ of ref. [32] is a deviation from (our) previous expectations [43, 44] for the small-x expansion. We have discussed this issue already in ref. [33]; here we address it in a different (but related, of course) manner.

Transforming the coefficients $\gamma_{u}^{(n)a}$ in eq. (2.4), which we have provided in an ancillary file to ref. [33], to x-space and expanding in powers of x and $\ln x$,

$$P_{\rm u}^{(n)a}(x) \;=\; \sum_{p=0} \sum_{\ell=0}^{2n} x^p \ln^{2n-\ell} x \; \mathbb{P}_{n,a}^{(p,\ell)}\;, \tag{2.7}$$

one arrives at the double-log (DL, $\ell < n$) coefficients of $P_{\rm u}^{+}(x)$ for the first four values of p:

$$\begin{aligned}
\mathbb{P}_{1,+}^{(0,0)} &= 0\\
\mathbb{P}_{1,+}^{(1,0)} &= -8\,C_F C_{FA}\\
\mathbb{P}_{1,+}^{(2,0)} &= 0\\
\mathbb{P}_{1,+}^{(3,0)} &= -16\,C_F C_{FA}\;,
\end{aligned} \tag{2.8}$$

PoS(LL2026)022

$$
\begin{aligned}
\mathbb{P}_{2,+}^{(0,0)} &= \mathbb{P}_{2,+}^{(0,1)} = 0 \\
\mathbb{P}_{2,+}^{(1,0)} &= -4\,C_F C_{FA}^2 + \frac{16}{3}\,C_F^2 C_{FA} \\
\mathbb{P}_{2,+}^{(1,1)} &= -\frac{368}{9}\,C_F C_{FA}^2 - \frac{16}{9}\,C_F n_f C_{FA} + \frac{440}{9}\,C_F^2 C_{FA} \\
\mathbb{P}_{2,+}^{(2,0)} &= \mathbb{P}_{2,+}^{(2,1)} = 0 \\
\mathbb{P}_{2,+}^{(3,0)} &= -\frac{32}{3}\,C_F C_{FA}^2 + 16\,C_F^2 C_{FA} \\
\mathbb{P}_{2,+}^{(3,1)} &= -\frac{1696}{9}\,C_F C_{FA}^2 - \frac{32}{9}\,C_F n_f C_{FA} + 128\,C_F^2 C_{FA}\,,
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
\mathbb{P}_{3,+}^{(0,0)} &= \mathbb{P}_{3,+}^{(0,1)} = 0 \\
\mathbb{P}_{3,+}^{(0,2)} &= \zeta_2\left(-48\,D_{FA} - 288\,C_F C_{FA}^3 + 384\,C_F^2 C_{FA}^2 - 112\,C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,+}^{(1,0)} &= -\frac{32}{15}\,C_F C_{FA}^3 + \frac{152}{45}\,C_F^2 C_{FA}^2 - \frac{92}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,+}^{(1,1)} &= -\frac{376}{15}\,C_F C_{FA}^3 - \frac{8}{15}\,C_F n_f C_{FA}^2 + \frac{2252}{45}\,C_F^2 C_{FA}^2 + \frac{8}{9}\,C_F^2 n_f C_{FA} - \frac{1448}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,+}^{(1,2)} &= -\frac{32}{3}\,D_{FA} + \frac{2992}{27}\,C_F C_{FA}^3 - \frac{104}{27}\,C_F n_f C_{FA}^2 - \frac{8}{27}\,C_F n_f^2 C_{FA} - \frac{2984}{27}\,C_F^2 C_{FA}^2 \\
&\quad + \frac{128}{27}\,C_F^2 n_f C_{FA} + \frac{2836}{27}\,C_F^3 C_{FA} + \zeta_2\left(16 D_{FA} + \frac{704}{3}\left(C_F C_{FA}^3 - C_F^2 C_{FA}^2\right) + 80 C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,+}^{(2,0)} &= \mathbb{P}_{3,+}^{(2,1)} = 0 \\
\mathbb{P}_{3,+}^{(2,2)} &= \zeta_2\left(-128\,D_{FA} - 704\,C_F C_{FA}^3 + 928\,C_F^2 C_{FA}^2 - 288\,C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,+}^{(3,0)} &= -\frac{64}{9}\,C_F C_{FA}^3 + \frac{32}{3}\,C_F^2 C_{FA}^2 - \frac{64}{9}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,+}^{(3,1)} &= -\frac{160}{9}\,D_{FA} - \frac{4768}{45}\,C_F C_{FA}^3 - \frac{64}{45}\,C_F n_f C_{FA}^2 + \frac{10352}{45}\,C_F^2 C_{FA}^2 + \frac{128}{45}\,C_F^2 n_f C_{FA} \\
&\quad - \frac{4624}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,+}^{(3,2)} &= -\frac{8260}{27}\,D_{FA} - \frac{544}{27}\,n_f D_{FF} + \frac{5248}{9}\,C_F C_{FA}^3 + \frac{208}{27}\,C_F n_f C_{FA}^2 - \frac{16}{27}\,C_F n_f^2 C_{FA} \\
&\quad - \frac{2696}{9}\,C_F^2 C_{FA}^2 - \frac{208}{9}\,C_F^2 n_f C_{FA} + \frac{11744}{27}\,C_F^3 C_{FA} \\
&\quad + \zeta_2\left(\frac{1472}{3}\,C_F C_{FA}^3 - \frac{1568}{3}\,C_F^2 C_{FA}^2 + 224\,C_F^3 C_{FA}\right)\,.
\end{aligned}
\tag{2.10}
$$

Here and below we have used the abbreviations $C_{FA} = C_F - 1/2\,C_A$ and

$$
D_{FF} = \frac{d_F^{abcd} d_F^{abcd}}{n_c} - \frac{1}{48}\,C_A^2 C_F\,, \quad D_{FA} = \frac{d_F^{abcd} d_A^{abcd}}{n_c} - \frac{1}{24}\,C_A^3 C_F \tag{2.11}
$$

for large-n_c suppressed combinations of group invariants.

The coefficients of the even powers p of x in eqs. (2.8), (2.9) vanish; the odd-p coefficients are non-zero already at the leading-log (LL, $\ell = 0$) level, except in the large-n_c (Ln_c) limit of SU(n_c). The surprising (to us) result of ref. [32] is that this even-p pattern does not hold at four loops ($n = 3$). Non-vanishing $\zeta_2 = \pi^2/6$ next-to-next-to-leading log (NNLL, $\ell = 2$) coefficients occur for all n_f^0

PoS(LL2026)022

colour factors; the $\ell = 2$ contribution only vanishes in the Ln_c limit. Consequently the even-p $x^p \ln^4 x$ terms of $P_{\rm ns}^{(3)+}(x)$ are not fixed by lower-order results via eq. (2.4), and the x^0-predictions of refs. [43, 44] agree, for the π^2 terms, only at large n_c with the result of ref. [32].

The corresponding coefficients for $P_{\rm u}^-(x)$ in the small-x expansion (2.7) are given by

$$\begin{aligned}
\mathbb{P}_{1,-}^{(0,0)} &= -8\,C_F C_{FA} \\
\mathbb{P}_{1,-}^{(1,0)} &= 0 \\
\mathbb{P}_{1,-}^{(2,0)} &= -16\,C_F C_{FA} \\
\mathbb{P}_{1,-}^{(3,0)} &= 0\,,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\mathbb{P}_{2,-}^{(0,0)} &= -4\,C_F C_{FA}^2 + \frac{16}{3}\,C_F^2 C_{FA} \\
\mathbb{P}_{2,-}^{(0,1)} &= -\frac{368}{9}\,C_F C_{FA}^2 - \frac{16}{9}\,C_F n_f C_{FA} + \frac{440}{9}\,C_F^2 C_{FA} \\
\mathbb{P}_{2,-}^{(1,0)} &= \mathbb{P}_{2,-}^{(1,1)} = 0 \\
\mathbb{P}_{2,-}^{(2,0)} &= -\frac{32}{3}\,C_F C_{FA}^2 + 16\,C_F^2 C_{FA} \\
\mathbb{P}_{2,-}^{(2,1)} &= -\frac{928}{9}\,C_F C_{FA}^2 - \frac{32}{9}\,C_F n_f C_{FA} + \frac{640}{9}\,C_F^2 C_{FA} \\
\mathbb{P}_{2,-}^{(3,0)} &= \mathbb{P}_{2,-}^{(3,1)} = 0\,,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\mathbb{P}_{3,-}^{(0,0)} &= -\frac{32}{15}\,C_F C_{FA}^3 + \frac{152}{45}\,C_F^2 C_{FA}^2 - \frac{92}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,-}^{(0,1)} &= -\frac{376}{15}\,C_F C_{FA}^3 - \frac{8}{15}\,C_F n_f C_{FA}^2 + \frac{2252}{45}\,C_F^2 C_{FA}^2 + \frac{8}{9}\,C_F^2 n_f C_{FA} - \frac{1448}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,-}^{(0,2)} &= -\frac{272}{27}\,C_F C_{FA}^3 + \frac{280}{27}\,C_F n_f C_{FA}^2 - \frac{8}{27}\,C_F n_f^2 C_{FA} - \frac{3032}{27}\,C_F^2 C_{FA}^2 - \frac{448}{27}\,C_F^2 n_f C_{FA} \\
&\quad + \frac{1396}{27}\,C_F^3 C_{FA} + \zeta_2 \left(16 D_{FA} + \frac{704}{3}\,C_F C_{FA}^3 - \frac{704}{3}\,C_F^2 C_{FA}^2 + 80 C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,-}^{(1,0)} &= \mathbb{P}_{3,-}^{(1,1)} = 0 \\
\mathbb{P}_{3,-}^{(1,2)} &= \zeta_2 \left(-48\,D_{FA} - 288\,C_F C_{FA}^3 + 384\,C_F^2 C_{FA}^2 - 112\,C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,-}^{(2,0)} &= -\frac{64}{9}\,C_F C_{FA}^3 + \frac{32}{3}\,C_F^2 C_{FA}^2 - \frac{64}{9}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,-}^{(2,1)} &= -\frac{32}{5}\,D_{FA} - \frac{544}{9}\,C_F C_{FA}^3 - \frac{64}{45}\,C_F n_f C_{FA}^2 + \frac{400}{3}\,C_F^2 C_{FA}^2 + \frac{128}{45}\,C_F^2 n_f C_{FA} - \frac{2704}{45}\,C_F^3 C_{FA} \\
\mathbb{P}_{3,-}^{(2,2)} &= -\frac{292}{3}\,D_{FA} - \frac{32}{3}\,n_f D_{FF} + \frac{18560}{27}\,C_F C_{FA}^3 + \frac{464}{27}\,C_F n_f C_{FA}^2 - \frac{16}{27}\,C_F n_f^2 C_{FA} \\
&\quad - \frac{27352}{27}\,C_F^2 C_{FA}^2 - \frac{944}{27}\,C_F^2 n_f C_{FA} + \frac{18080}{27}\,C_F^3 C_{FA} \\
&\quad + \zeta_2 \left(\frac{1472}{3}\,C_F C_{FA}^3 - \frac{1568}{3}\,C_F^2 C_{FA}^2 + 224 C_F^3 C_{FA}\right) \\
\mathbb{P}_{3,-}^{(3,0)} &= \mathbb{P}_{3,-}^{(3,1)} = 0 \\
\mathbb{P}_{3,-}^{(3,2)} &= \zeta_2 \left(-128\,D_{FA} - 704\,C_F C_{FA}^3 + 928\,C_F^2 C_{FA}^2 - 288\,C_F^3 C_{FA}\right)\,.
\end{aligned} \tag{2.14}$$

PoS(LL2026)022

The structure is the same as in eqs. (2.8) – (2.10), except that the roles of the even and odd powers of p are interchanged. The pattern for $P_{\rm u}^{\rm s}(x)$, also based on odd-N moments in the OPE (unlike $P_{\rm u}^{+}(x)$, see eq. (1.4) above) is – up to some additional zeros – as in eqs. (2.12) – (2.14) with

$$\begin{aligned}
\mathbb{P}_{2,\rm s}^{(0,0)} &= \frac{16}{3}\,n_f\,\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{2,\rm s}^{(0,1)} &= -\frac{32}{3}\,n_f\,\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{2,\rm s}^{(1,0)} &= \mathbb{P}_{2,\rm s}^{(1,1)} = 0 \\
\mathbb{P}_{2,\rm s}^{(2,0)} &= 0 \\
\mathbb{P}_{2,\rm s}^{(2,1)} &= -\frac{256}{9}\,n_f\,\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{2,\rm s}^{(3,0)} &= \mathbb{P}_{2,\rm s}^{(3,1)} = 0\,,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\mathbb{P}_{3,\rm s}^{(0,0)} &= -\frac{16}{5}\,C_A n_f\,\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(0,1)} &= \left(\frac{32}{15}\,n_f^2 - \frac{208}{15}\,C_A n_f + \frac{32}{15}\,C_F n_f\right)\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(0,2)} &= \left(-\frac{256}{9}\,n_f^2 - \frac{1760}{9}\,C_A n_f + \frac{112}{3}\,C_F n_f + \zeta_2\left(144\,C_A n_f - \frac{32}{3}\,C_F n_f\right)\right)\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(1,0)} &= \mathbb{P}_{3,\rm s}^{(1,1)} = 0 \\
\mathbb{P}_{3,\rm s}^{(1,2)} &= \zeta_2\left(48\,C_A n_f - 32\,C_F n_f\right)\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(2,0)} &= 0 \\
\mathbb{P}_{3,\rm s}^{(2,1)} &= \frac{256}{15}\,C_A n_f\,\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(2,2)} &= \left(-\frac{128}{9}\,n_f^2 - \frac{3520}{27}\,C_A n_f + \frac{832}{27}\,C_F n_f\right)\frac{d^{abc}d_{abc}}{n_c} \\
\mathbb{P}_{3,\rm s}^{(3,0)} &= \mathbb{P}_{3,\rm s}^{(3,1)} = 0 \\
\mathbb{P}_{3,\rm s}^{(3,2)} &= 0\,.
\end{aligned} \tag{2.16}$$

The ζ_2 terms that fall outside the expected structure are $\mathbb{P}_{3,+}^{(0,2)}$ and $\mathbb{P}_{3,+}^{(2,2)}$ in eq. (2.10), $\mathbb{P}_{3,-}^{(1,2)}$ and $\mathbb{P}_{3,-}^{(3,2)}$ in eq. (2.14) and $\mathbb{P}_{3,\rm s}^{(1,2)}$ in eq. (2.16). The first four of these contribute to eq. (3.15) of ref. [33], which was derived not via the universal kernels, but following the approach of ref. [44] for the D-dimensional structure of unfactorized structure functions. As mentioned in ref. [33], these additional terms may be related to the issues studied in refs. [52, 53] and references therein.

The remaining non-vanishing N^{2}LO and N^{3}LO double-logarithmic coefficient in eqs. (2.9) – (2.16) – which all vanish in the large-n_c limit – have not been obtained from lower-order information either, with the exception of the leading-log coefficients $\mathbb{P}_{2,-}^{(0,0)}$ and $\mathbb{P}_{3,-}^{(0,0)}$ which were correctly predicted about 30 years ago [54, 55] on the basis of ref. [56]. Maybe the results collected above can be useful in future research towards improving upon this situation.

PoS(LL2026)022

For the rest of this subsection, we focus on the small-x expansion of the (reciprocity-respecting) n_f^0 quartic Casimir contribution, which does not include x^0 terms enhanced by $\ln^\ell x$ without π^2,

$$P_{\rm ns}^{(3)+}(x)\big|_{d_F^{abcd}d_A^{abcd}/n_c} \;=\; -48\,\zeta_2\ln^4 x \,-\, 256\,\zeta_2\ln^3 x \,-\left(2112\,\zeta_2 - 768\,\zeta_2^2\right)\ln^2 x \,+\,\ldots\;. \tag{2.17}$$

The $\ln^1 x$ term can be found in eq. (3.15) of ref. [33]. The only combinations of η in eq. (2.5) and binomial sums in eq. (2.6) to overall weight 7 that contribute to the terms shown in eq. (2.17) are

$$\eta^6\,\mathbb{S}_1\,,\quad \eta^5\big\{\mathbb{S}_1,\,\mathbb{S}_{1,1},\,\mathbb{S}_2\big\}\quad\text{and}\quad \eta^4\big\{\mathbb{S}_1,\,\mathbb{S}_2,\,\mathbb{S}_{1,1,1},\,\mathbb{S}_{1,2},\,\mathbb{S}_{2,1},\,\mathbb{S}_3\big\}\,. \tag{2.18}$$

Note that the even-N expression $\gamma_{\rm ns}^{(3)+}(N)$ does not include any terms with π^2. The part of this expression, completely given in eq. (C.1) of ref. [33], that leads to eq. (2.17) is

$$\begin{aligned}\frac{1}{16}\,\gamma_{\rm ns}^{(3)+}(N)\big|_{d_F^{abcd}d_A^{abcd}/n_c} \;=\;& \left(216\,\eta^4 + 168\,\eta^5 + 36\,\eta^6\right)\mathbb{S}_1 + \left(88\,\eta^4 + 24\,\eta^5\right)\mathbb{S}_2 \\ & - 20\,\eta^5\,\mathbb{S}_{1,1} + 14\,\eta^4\,\mathbb{S}_{1,2} - 10\,\eta^4\,\mathbb{S}_{2,1} + 40\,\eta^4\,\mathbb{S}_3 + \ldots\;.\end{aligned} \tag{2.19}$$

Here the coefficients in the first line are uniquely connected to the ζ_2^1 part and the absence of ζ_3 in eq. (2.17), those in the second line all contribute only to the coefficient of ζ_2^2.

2.3 Numerical results

During the past years, we have extended the fixed-N Forcer [45] computations of $P_{\rm ns}^{(3)\pm}$ in ref. [12] to $N = 22$ for all n_f^0 terms and to $N = 24$ of higher for the various n_f^1 contributions. The latter results were used to obtain the all-N expressions in ref. [29], the former – supplemented by the exact result for the four-loop cusp anomalous dimension $A_{\rm q}^{(4)}$ [46, 47] – to improve upon the approximations in ref. [12]. Of course, these newer approximations have now been superseded.

It is, however, not entirely uninteresting to compare them with the exact results [32], in particular in the case of $P_{\rm ns}^{(3)+}(x)$, where the exact result is slightly above the 2017 error band at $x < 10^{-3}$. This is done in the left panel of fig. 1. The additional information available in 2025, three more even-N or odd-N moments and the exact leading large-x contribution $A_{\rm q}^{(4)}/(1-x)_+$, lead to small uncertainties for about one additional order of magnitude in x towards $x = 0$.

The effects of the three-loop and four-loop splitting functions on the logarithmic derivatives $\dot{q}_{\rm ns}^{\,a} \equiv d\ln q_{\rm ns}^a/d\ln\mu^2$, $a = \pm, {\rm v}$, are illustrated in the right panel of fig. 1 and in fig. 2 for the default choice $\mu \equiv \mu_f = \mu_r$ of the factorization and renormalization scales already indicated in eq. (1.1). In all three cases we employ, as already in ref. [1], the same schematic model distribution

$$x q_{\rm ns}^{\pm,{\rm v}}(x,\mu_0^2) \;=\; x^{0.5}(1-x)^3 \tag{2.20}$$

and an order-independent value

$$\alpha_{\rm s}(\mu_0^2) \;=\; 0.2 \tag{2.21}$$

for the strong coupling constant which corresponds to a scale μ_0^2 in the range $25\ldots50\;{\rm GeV}^2$. This facilitates a direct comparison of effects of the various contributions to the splitting functions.

For the scale dependence of the flavour asymmetries $q_{\rm ns}^{\pm}$, the relative N^{3}LO corrections are well below 1% over the full range shown in the figures, except very close to the sign changes at $x \approx 0.07$. This represents a clear accuracy improvement over the corresponding N^{2}LO results –

PoS(LL2026)022

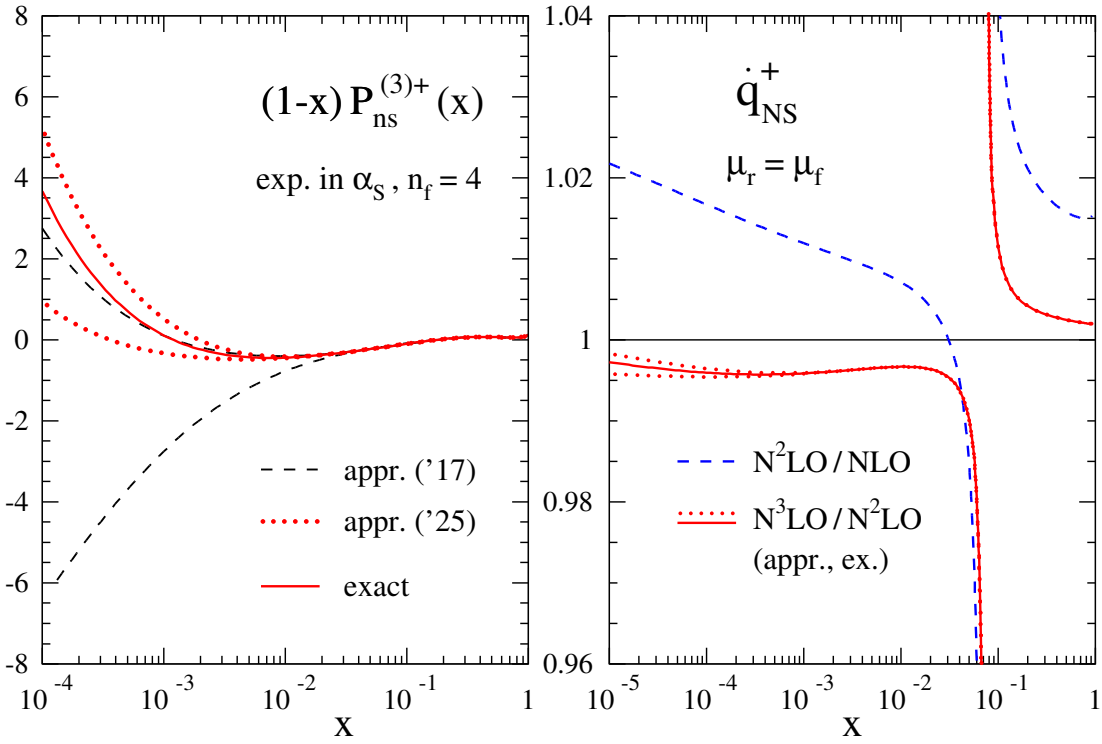

Figure 1: Left part: The new result [32] for the even-N based splitting function $P_{\rm ns}^{(3)+}$ for four light flavours, compared to our earlier approximations based on $N \le 16$ ('17) [12] and $N \le 22$ ('25) [unpubl.]. Right part: The relative N^{2}LO and N^{3}LO contributions to the scale derivative of $q_{\rm ns}^+$ for the input (2.20) and (2.21).

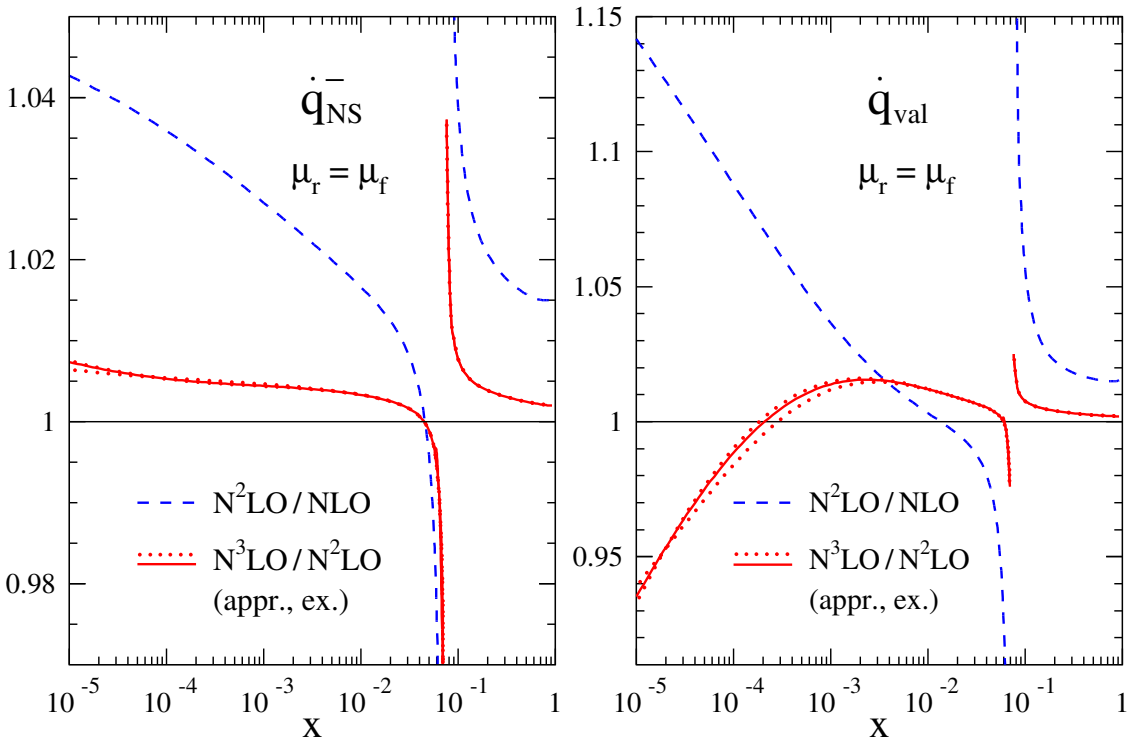

Figure 2: As the right part of fig. 1, but for the expansion of the splitting functions $P_{\rm ns}^-$ (left) and $P_{\rm ns}^{\rm v}$ (right).

PoS(LL2026)022

for the much larger NLO effects the reader is referred to figs. 6 and 7 of ref. [1]. The situation is different for the total valence distribution $q_{\rm val} \equiv q_{\rm ns}^{\rm v}$ at small x, due to the effects of the additional contributions $P_{\rm ns}^{(n)\rm s}$ at $n = 2$ and $n = 3$. These are suppressed at large x, but much larger than their $P_{\rm ns}^-$ counterparts at small x, see fig. 5 of ref. [1] and fig. 3 of ref. [32].

We have first used the FORTRAN code of ref. [48] for the numerical evaluation of harmonic polylogarithms (HPLs) $H_{\vec{w}}(x)$ [49], extended to weight-6 [50] for use in ref. [12], for the numerical evaluation of the exact results for $P_{\rm ns}^{(3)a}(x)$. Using its numerical output, we have then extended our parametrizations from the fermionic and large-n_c contributions [11, 12, 29] to the complete non-singlet splitting functions [32]. Using the abbreviations

$$\mathcal{D}_0 \,=\, 1/(1-x)_+\,, \quad x_1 \,=\, 1-x\,, \quad L_1 \,=\, \ln(1-x)\,, \quad L_0 \,=\, \ln x\,, \tag{2.22}$$

the N^{3}LO contribution to $P_{\rm ns}^+(x)$ in QCD can be approximated by

$$\begin{aligned}
P_{\rm ns}^{(3)+}(x) \;=\;
&\left\{2.070235\cdot 10^4 - 5.171916\cdot 10^3\, n_f + 1.955772\cdot 10^2\, n_f^2 + 3.272344\cdot 10^0\, n_f^3\right\}\mathcal{D}_0\\
&+\left\{23393.508 - 5550.044\, n_f + 193.8591\, n_f^2 + 3.014982\, n_f^3\right\}\delta(x_1)\\
&+25000\,\Big(x_1\big(3.8985 + 2.5079\,x - 0.56226\,x^2 + 0.41604\,x^3\big) - 1.8664\,x_1 L_1\\
&-1.8636\,L_0 L_1 + 0.17547\,x_1 L_1^2 + 0.5684\,x L_0 + 0.63438\,x L_0^2 + 0.06534\,x L_0^3\Big)\\
&+5.374286\cdot 10^4\, L_0 + 1.580664\cdot 10^4\, L_0^2 + 2.641883\cdot 10^3\, L_0^3 + 2.698936\cdot 10^2\, L_0^4\\
&+1.316872\cdot 10^1\, L_0^5 + 3.511660\cdot 10^{-1}\, L_0^6 - 3.1828934\cdot 10^4 + 1.6950937\cdot 10^4\, L_1\\
&+n_f\Big\{25000\,\Big(x_1\big(-0.74096 + 0.59892\,x - 0.15132\,x^2 + 0.10925\,x^3\big) + 1.0988\,x_1 L_1\\
&+1.1103\,L_0 L_1 - 0.01060\,x_1 L_1^2 + 1.0576\,x L_0 + 0.02492\,x L_0^2 + 0.00670\,x L_0^3\Big)\\
&-8.700346\cdot 10^3\, L_0 - 2.658355\cdot 10^3\, L_0^2 - 3.891653\cdot 10^2\, L_0^3\\
&-3.134156\cdot 10^1\, L_0^4 - 1.053498\, L_0^5 + 8.420314\cdot 10^3 - 2.741830\cdot 10^3\, L_1\Big\}\\
&+n_f^2\Big\{250\,\Big(x_1\big(3.0008 + 0.8619\,x - 0.12411\,x^2 + 0.31595\,x^3\big) - 0.37529\,x L_0\\
&-0.21684\,x\,L_0^2 - 0.02295\,x L_0^3 + 0.03394\,x_1 L_1 + 0.40431\,L_0 L_1\Big)\\
&+3.930056\cdot 10^2\, L_0 + 1.125705\cdot 10^2\, L_0^2 + 1.652675\cdot 10^1\, L_0^3\\
&+7.901235\cdot 10^{-1}\, L_0^4 - 3.760092\cdot 10^2 + 2.668861\cdot 10^1\, L_1\Big\}\\
&+n_f^3\Big\{-2.426296 - 0.8460488\,x + \big(0.5267490\,x_1^{-1} - 3.687243 + 3.160494\,x\big)\,L_0\\
&-\big(1.316872\,(x_1^{-1}+0.1) - 1.448560\,x\big)\,L_0^2 - \big(0.2633745\,x_1^{-1} - 0.131687\,(1+x)\big)\,L_0^3\Big\}\,.
\end{aligned} \tag{2.23}$$

The n_f^3 part, which is the Mellin inverse of an old leading large-n_f result in N-space [51], is exact up to the rounding of the coefficients. In all other parts, the coefficients of the form $a\cdot 10^b$ are exact up to a rounding of a, while the five-digit coefficients in the larger round brackets have been fitted to the exact values of $P_{\rm ns}^{(3)+}(x)$ in the range $10^{-6} \le x \le 1 - 10^{-6}$. Finally the coefficients of $\delta(1-x)$ in the second line of eq. (2.23) have been slightly adjusted from their exact values [32], by -0.282, -0.001 and 0.0037 for the n_f^0, n_f^1 and n_f^2 contributions, for a maximal accuracy of the

PoS(LL2026)022

second moment and convolutions with the quark distributions.

A corresponding accurate parametrization for $P_{\rm ns}^-(x)$ is given by

$$
\begin{aligned}
P_{\rm ns}^{(3)-}(x) \;=\;
&\left\{ 2.070235 \cdot 10^4 - 5.171916 \cdot 10^3\, n_f + 1.955772 \cdot 10^2\, n_f^2 + 3.272344 \cdot 10^0\, n_f^3 \right\} \mathcal{D}_0 \\
&+ \left\{ 23393.102 - 5550.030\, n_f + 193.8559\, n_f^2 + 3.014982\, n_f^3 \right\} \delta(x_1) \\
&+ 25000 \Big(x_1 \left(4.0424 + 2.2915\, x - 0.90975\, x^2 + 0.84854\, x^3 \right) - 0.81082\, x_1 L_1 \\
&- 0.88065\, L_0 L_1 + 0.19038\, x_1 L_1^2 + 0.43116\, x L_0 + 0.12820\, x L_0^2 + 0.00391\, x L_0^3 \Big) \\
&+ 5.615655 \cdot 10^4\, L_0 + 1.970952 \cdot 10^4\, L_0^2 + 3.213435 \cdot 10^3\, L_0^3 + 2.981618 \cdot 10^2\, L_0^4 \\
&+ 1.851523 \cdot 10^1\, L_0^5 + 9.964335 \cdot 10^{-1}\, L_0^6 - 3.1828934 \cdot 10^4 + 1.6950937 \cdot 10^4\, L_1 \\
&+ n_f \Big\{ 25000 \Big(x_1 \left(-0.80350 + 0.72278\, x - 0.16905\, x^2 + 0.11719\, x^3 \right) + 1.3324\, x_1 L_1 \\
&+ 1.3990\, L_0 L_1 - 0.01710\, x_1 L_1^2 + 1.2984\, x L_0 + 0.00738\, x\, L_0^2 + 0.00487\, x L_0^3 \Big) \\
&- 1.023783 \cdot 10^4\, L_0 - 3.297010 \cdot 10^3\, L_0^2 - 4.709831 \cdot 10^2\, L_0^3 - 3.460082 \cdot 10^1\, L_0^4 \\
&- 9.744856 \cdot 10^{-1}\, L_0^5 + 8.420314 \cdot 10^3 - 2.741830 \cdot 10^3\, L_1 \Big\} \\
&+ n_f^2 \Big\{ 250 \Big(x_1 \left(3.2206 + 1.7507\, x + 0.13281\, x^2 + 0.45969\, x^3 \right) + 1.5641\, x L_0 \\
&- 0.37902\, x\, L_0^2 - 0.03248\, x L_0^3 + 2.7511\, x_1 L_1 + 3.2709\, L_0 L_1 \Big) \\
&+ 4.378810 \cdot 10^2\, L_0 + 1.282948 \cdot 10^2\, L_0^2 + 1.959945 \cdot 10^1\, L_0^3 \\
&+ 9.876543 \cdot 10^{-1}\, L_0^4 - 3.760092 \cdot 10^2 + 2.668861 \cdot 10^1\, L_1 \Big\} \qquad (2.24) \\
&+ n_f^3 \Big\{ -2.426296 - 0.8460488\, x + \left(0.5267490\, x_1^{-1} - 3.687243 + 3.160494\, x \right) L_0 \\
&- \left(1.316872\, (x_1^{-1} + 0.1) - 1.448560\, x \right) L_0^2 - \left(0.2633745\, x_1^{-1} - 0.131687\, (1 + x) \right) L_0^3 \Big\} \,,
\end{aligned}
$$

where the n_f^3 part is the same as in eq. (2.23), and the $\delta(1-x)$ adjustments, fixed by the numerical determination of the first moment, are −0.688, −0.015 and 0.0095.

Finally the four-loop contribution to $P_{\rm ns}^{\rm s}(x)$ can be approximated by

$$
\begin{aligned}
P_{\rm ns}^{(3)\rm s}(x) \;=\;
&n_f \Big\{ 250 \Big(x_1 \left(34.060 - 0.84735\, x - 6.4973\, x^2 + 1.1689\, x^3 - 0.32905\, x L_0 \right) \\
&- 4.6473\, x L_0^2 - 1.7459\, x L_0^3 - x_1^2\, L_1 \left(1.2384 + 0.13345\, L_1 \right) \Big) + 7.729670 \cdot 10^3\, L_0 \\
&+ 2.306325 \cdot 10^3\, L_0^2 - 1.629627 \cdot 10^2\, L_0^3 - 1.694370 \cdot 10^1\, L_0^4 + 1.580247\, L_0^5 \\
&- 1.876543\, L_0^6 - 2.743332 \cdot 10^2\, x_1 L_1 + 1.164936 \cdot 10^{-1}\, x_1 L_1^2 \Big\} \\
&+ n_f^2 \Big\{ 250 \Big(x_1 \left(-4.7702 + 1.7922\, x + 0.64510\, x^2 - 0.15489\, x^3 - 0.64749\, x\, L_0 \right) \\
&+ 0.83597\, x L_0^2 + 0.10964\, x L_0^3 + x_1^2 L_1 \left(0.03048 + 0.02146\, L_1 \right) \Big) - 6.473971 \cdot 10^2\, L_0 \\
&- 6.641219 \cdot 10^1\, L_0^2 - 5.353347\, L_0^3 - 5.925926\, L_0^4 - 3.950617 \cdot 10^{-1}\, L_0^5 \\
&+ 1.970002 \cdot 10^1\, x_1 L_1 - 3.435474\, x_1 L_1^2 \Big\} \,. \qquad (2.25)
\end{aligned}
$$

Also here the small-x and large-x coefficients written in the form $a \cdot 10^b$ are exact up to a rounding of a to seven significant digits.

PoS(LL2026)022

3. The singlet cases

3.1 General properties

So far, all-N results for the N^{3}LO singlet anomalous dimensions have been obtained only for the n_f^3 terms, the complete n_f^2 contributions to $\gamma_{\rm ps}^{(3)}$ (pure singlet, see eq. (1.2)) and $\gamma_{\rm gq}^{(3)}$, and the $C_F^2 n_f^2$ part of $\gamma_{\rm gg}^{(3)}$ [11, 20, 22, 31]. In all cases, except for the n_f^0 parts of $\gamma_{\rm gq}^{(3)}$ and the n_f^1 parts of $\gamma_{\rm qg}^{(3)}$, the non-rational (ζ-function) contributions are completely known [13, 18, 19, 24, 27, 30, 31]. Already these results show several features that did either not occur at three loops and/or that transcend the structure (2.1) extended, as already for $\gamma_{\rm ns}^{\rm s}$, by terms with powers of $(N-1)^{-1}$ and $(N+2)^{-1}$.

All singlet anomalous dimensions except $\gamma_{\rm ps}^{(3)}$ – where terms with $k = 0$ in eq. (2.1) occur for the first time – include ζ-function terms with N in the numerator, in the form $(N-1)$ and $(N+2)$ for $\gamma_{\rm qg}^{(3)}$ and $\gamma_{\rm gq}^{(3)}$, respectively, and with $N(N+1)$ for $\gamma_{\rm gg}^{(3)}$. Where both the ζ_5 and ζ_3 terms are fully known, these numerator structures are multiplied by the ζ-function part of the function

$$\begin{aligned} f(N) &= 5\zeta_5 + 4\zeta_3 S_{-2} - 2S_{-5} - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5 \\ &= 5\zeta_5 - 2\zeta_3 \mathbb{S}_2 - \mathbb{S}_{2,1,2} + \mathbb{S}_{3,1,1} \,, \end{aligned} \tag{3.1}$$

where, as usual, the sums are evaluated at the argument N. $f(N)$ behaves as $1/N^2$ at large-N. This function was first encountered, also with positive powers on N, in the three-loop coefficient functions for inclusive DIS [3, 4]. As expected from the ζ_5 and ζ_3 terms known before, it occurs as $S_1^2 f(N)$ as the 'wrapping correction' [57] in the n_f^0 parts [32] of $\gamma_{\rm ns}^{(n)\pm}(N)$. There is a second combination of sums that occurs with $N(N+1)$ in $\gamma_{\rm gg}^{(3)}$, and without this prefactor in $\gamma_{\rm ps}^{(3)}$ [18, 31]:

$$\widetilde{f}(N) \;=\; S_3 - 2S_{-3} + 4S_{2,1} \quad \text{with} \quad \widetilde{f}(N) \to N^{-2}(2\ln N - 1/2) \quad \text{for} \quad N \to \infty \,. \tag{3.2}$$

Another structure that does not occur in the anomalous dimensions up to three loops [1, 2] (note the argument of the sum) is [20, 22]

$$x^{-2} H_{-1,0}(x) \;=\; x^{-2}\left[\ln(x)\ln(1+x) + {\rm Li}_2(-x)\right] \quad \Leftrightarrow \quad (N-2)^{-2} S_2(N-2) \,. \tag{3.3}$$

Its $N = 2$ moment is $-3/2\,\zeta_3$, hence it leads to a $\delta_{N,2}$ term in the analytical N-space expressions with ζ_3 occurring in all four-loop singlet anomalous dimensions, but not in the non-singlet cases. The function (3.3) does occur already in two-loop coefficient functions in DIS, first calculated in refs. [58, 59] and, incidentally, presented at what became the first in this series of meetings.

3.2 Numerical results

In refs. [18, 19, 24, 27] we have computed the even-N moments of all four four-loop singlet splitting functions (1.2) to $N = 20$ via operator matrix elements using Forcer [45], and used these moments, supplemented by small-x and large-x endpoint information [44, 46, 47, 60–67], to construct approximate x-space expressions that are sufficiently accurate over a rather wide range in x, but unavoidably suffer from large uncertainties at $x \lesssim 10^{-3}$. They have been validated, and for $P_{\rm qg}^{(3)}(x)$ slightly shifted, by comparing the resulting error band at $N = 22$ with the exact values in ref. [30]. The resulting uncertainty bands are shown in figs. 3 and 4 for four light flavours.

PoS(LL2026)022

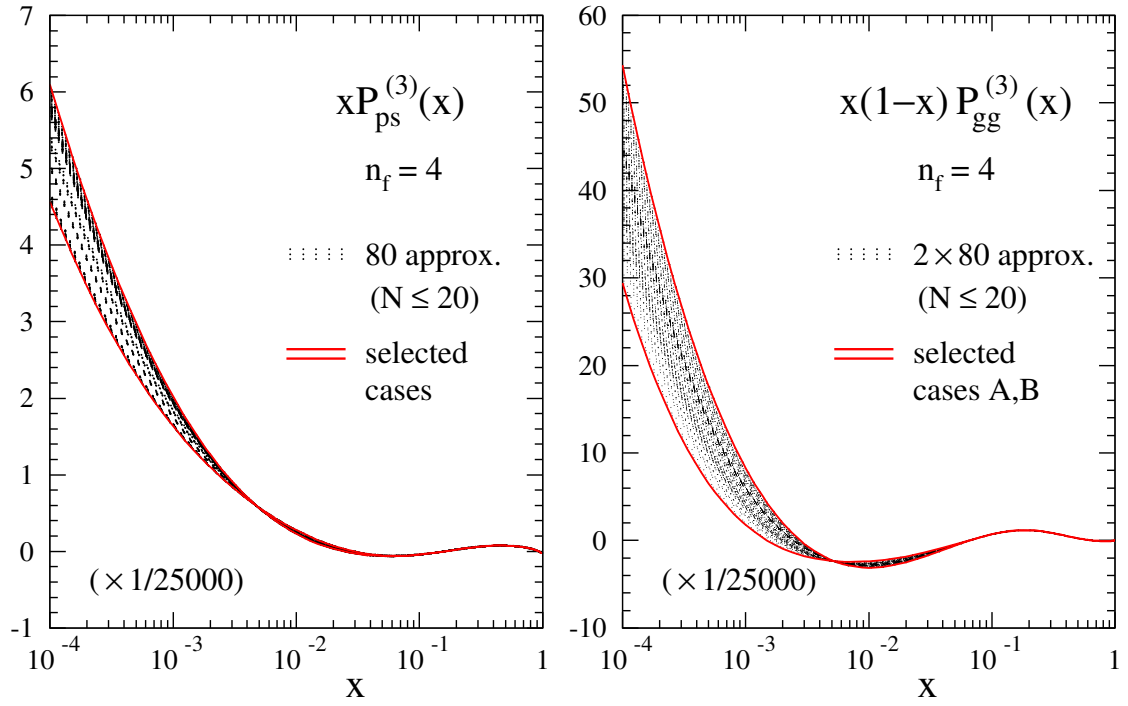

Figure 3: Approximations for the 4-loop contributions to the pure-singlet (ps) and gluon-gluon splitting functions in eq. (1.2) for $n_f = 4$ light flavours, as constructed in refs. [18, 27]. The curves have been divided by $25000 \simeq (4\pi)^4$, they thus correspond to an expansion in powers of α_s instead of $\alpha_s/(4\pi)$ as in eq. (1.3).

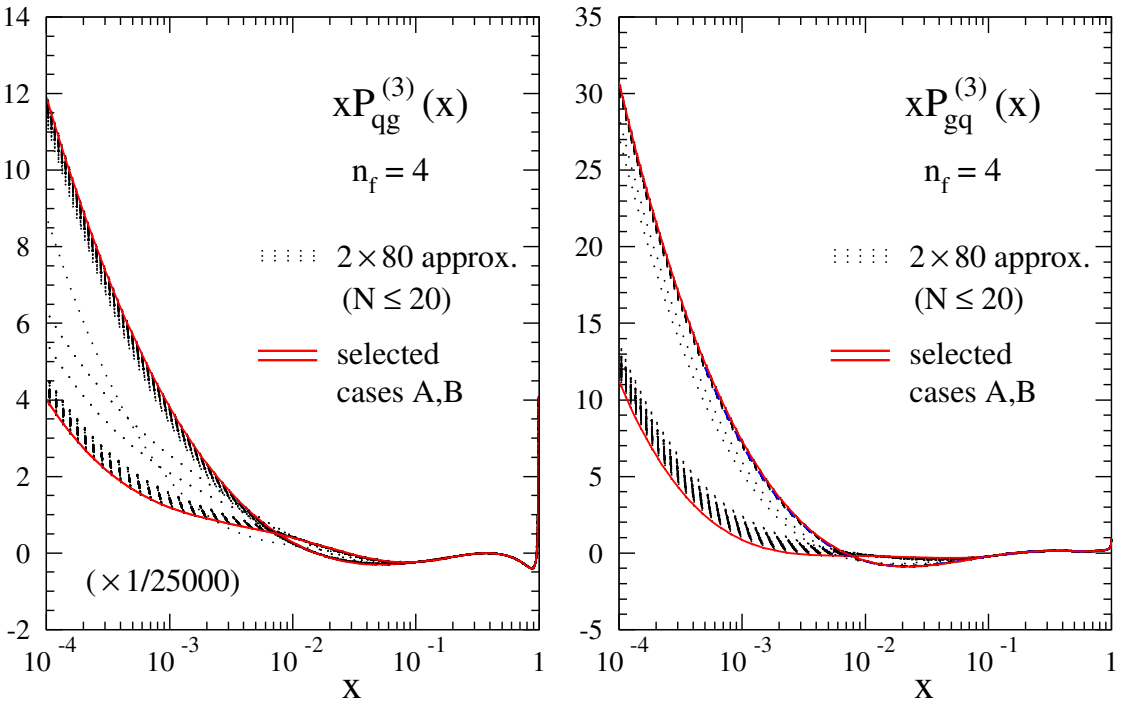

Figure 4: As fig. 3, but for the gluon-to-quark (qg) and quark-to-gluon (gq) splitting functions as approximated in refs. [19, 30].

PoS(LL2026)022

Due to the interplay of large-N constraints sampling high values of x in eq. (1.4), the smooth intermediate-x functions, and the small-x predictions, the latter depend on how well the large-x behaviour of the splitting functions is known beforehand. This situation is most favourable for pure-singlet case, with unknown coefficients of $(1-x)\ln^k x$, $k = 1, 2$, somewhat worse for the gluon-gluon case – before refs. [32, 33], due to a very small but not irrelevant residual uncertainty of the $\delta(1-x)$ contribution – and considerably worse for the off-diagonal cases, where the $(1-x)^0 \ln^{k'}(1-x)$ are covered by the present generalized large-x resummation only for $k' > 3$ [66, 67].

In order to illustrate the size and uncertainties of the present N^{3}LO contributions to the evolution of the singlet PDFs, we choose, as for the previous order in ref. [2], $\alpha_s(\mu_0^2) = 0.2$, $n_f = 4$, and the sufficiently realistic input distributions

$$\begin{aligned} xq_s(x,\mu_0^2) &= 0.6\,x^{-0.3}(1-x)^{3.5}\left(1+5.0\,x^{0.8}\right), \\ xg(x,\mu_0^2) &= 1.6\,x^{-0.3}(1-x)^{4.5}\left(1-0.6\,x^{0.3}\right). \end{aligned} \tag{3.4}$$

The resulting relative N^{2}LO and N^{3}LO effects on the scale derivative at this reference point are shown in fig. 5. Due to the Mellin convolutions with q_s and g in eq. (3.4), the uncertainties of the 4-loop splitting functions become relevant only at x-values below about 10^{-4}. The N^{3}LO corrections are generally small, reaching $(2 \pm 1)\%$ at $x = 10^{-5}$. The higher-order effect will be smaller at higher scales and larger at smaller scales.

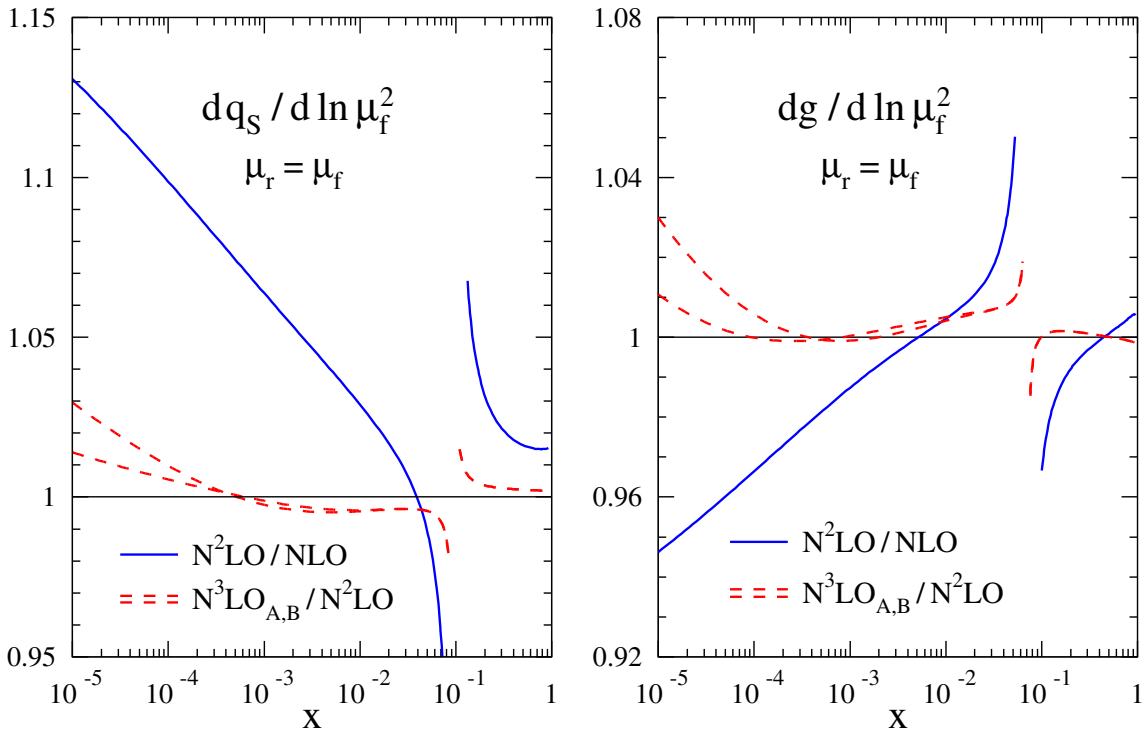

Figure 5: The relative N^{2}LO and N^{3}LO corrections to the scale derivatives of singlet quark (left panel) and gluon (right panel) PDFs at a standard reference scale μ_0 with $n_f = 4$ and $\alpha_s(\mu_0^2) = 0.2$, as obtained using the selected approximations in figs. 3 and 4. Note that that $dq_s/d\ln\mu^2$ and $dg/d\ln\mu^2$ change sign close to $x = 0.1$ and $x = 0.07$, respectively, which leads to irrelevant singularities in the relative corrections.

PoS(LL2026)022

4. Concluding remarks

The recently completed N^{3}LO non-singlet splitting functions $P_{\rm ns}^{(3)}$ [32] agree with all previous numerical and analytical expectations, see also ref. [33], except for next-to-next-to-leading small-x logarithms, where the n_f^0 terms with ζ_2 differ, beyond the limit of a large number of colours, from the predictions in ref. [44]. The new four-loop structure indicated by these differences and the single-log terms in the n_f^0 quartic-Casimir contributions may also affect related predictions for the coefficient functions in inclusive DIS [44] and related quantities in semi-inclusive e^+e^- annihilation [68].

The N^{3}LO singlet splitting functions are only known in an approximate form [30] which, however, should be sufficiently accurate for most practical applications. We expect that also these functions will be computed rather soon along the lines of ref. [32], hence large efforts for incremental improvements upon ref. [30] are not called for. In the absence of complete results, though, the determination of all $x^{-1}\ln x$ terms would be a game changer. In this context it seems remarkable that the 1994 paper [61] still appears to represent the state of the art for $P_{\rm ps,\,qg}^{(3)}$ at small x.

A FORM file of the expansion coefficients in eqs. (2.8) – (2.16) and FORTRAN files of the parametrizations (2.23) – (2.25), as well their exact counterparts, can be obtained from the preprint server http://arXiv.org, in a ready-to-use form analogous to the codes distributed with refs. [1, 2, 12]. They are also available, for now, from the authors upon request.

Acknowledgements

This work has been supported by the Deutsche Forschungsgemeinschaft (DFG), Research Unit FOR 2926, project number 409651613; the European Research Council (ERC) Advanced Grant 101095857; the UK Science and Technology Facilities Council (STFC), Consolidated Grant ST/X000699/1; and by a Research Award of the Alexander-von-Humboldt Foundation.

References

[1] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three loop splitting functions in QCD: the non-singlet case*, Nucl. Phys. B688 (2004) 101 [hep-ph/0403192]

[2] A. Vogt, S. Moch and J.A.M. Vermaseren, *The three-loop splitting functions in QCD: the singlet case* Nucl. Phys. B691 (2004) 129, [hep-ph/0404111]

[3] J.A.M. Vermaseren, A. Vogt and S. Moch, *The third-order QCD corrections to deep-inelastic scattering by photon exchange*, Nucl. Phys. B724 (2005) 3 [hep-ph/0504242]

[4] S. Moch, J.A.M. Vermaseren and A. Vogt, *Third-order QCD corrections to the charged-current structure function F_3*, Nucl. Phys. B813 (2009) 220 [arXiv:0812.4168]

[5] J. Currie et al, *N^3LO corrections to jet production in deep inelastic scattering using the Projection-to-Born method*, JHEP 05 (2018) 209 [arXiv:1803.09973]

[6] T. Gehrmann, A. Huss, J. Niehues, A. Vogt and D. M. Walker, *Jet production in charged-current deep-inelastic scattering to third order in QCD*, Phys. Lett. B792 (2019) 182 [arXiv:1803.06014]

[7] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, Phys. Rev. Lett. 114 (2015) 212001 [arXiv:1503.06056]

[8] B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, JHEP 05 (2018) 028 [arXiv:1802.00833]

PoS(LL2026)022

[9] X. Chen et al, *Fully Differential Higgs Boson Production to Third Order in QCD*, Phys. Rev. Lett. 127 (2021) 072002 [arXiv:2102.07607]

[10] B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt, *First Forcer results on deep-inelastic scattering and related quantities*, PoS LL2016 (2016) 071 [arXiv:1605.08408]

[11] J. Davies, A. Vogt, B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Large-n_f contributions to the four-loop splitting functions in QCD*, Nucl. Phys. B915 (2017) 335 [arXiv:1610.07477]

[12] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren, A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, JHEP 10 (2017) 041 [arXiv:1707.08315]

[13] J. Davies and A. Vogt, *Absence of π^2 terms in physical anomalous dimensions in DIS: Verification and resulting predictions*, Phys. Lett. B776 (2018) 189 [arXiv:1711.05267]

[14] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, Phys. Lett. B782 (2018) 627 [arXiv:1805.09638]

[15] A. Vogt, F. Herzog, S. Moch, B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Anomalous dimensions and splitting functions beyond the next-to-next-to-leading order*, PoS LL2018 (2018) 050 [arXiv:1808.08981]

[16] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Low moments of the four-loop splitting functions in QCD*, Phys. Lett. B825 (2022) 136853 [arXiv:2111.15561]

[17] G. Falcioni and F. Herzog, *Renormalization of gluonic leading-twist operators in covariant gauges*, JHEP 05 (2022) 177 [arXiv:2203.11181]

[18] G. Falcioni, F. Herzog, S. Moch and A. Vogt, *Four-loop splitting functions in QCD - The quark-quark case* , Phys. Lett. B842 (2023) 137944 [arXiv:2302.07593]

[19] G. Falcioni, F. Herzog, S. Moch and A. Vogt, *Four-loop splitting functions in QCD – The gluon-to-quark case*, Phys. Lett. B846 (2023) 138215 [arXiv:2307.04158]

[20] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.Z. Yang, *Complete n_f^2 contributions to four-loop pure-singlet splitting functions*, JHEP 01 (2024) 029 [arXiv:2308.07958]

[21] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.Z. Yang, *The $n_f C_F^3$ contribution to the non-singlet splitting function at four-loop order*, Phys. Lett. B849 (2024) 138427 [2310.12240]

[22] G. Falcioni, F. Herzog, S. Moch, J.A.M Vermaseren and A. Vogt, *The double fermionic contribution to the four-loop quark-to-gluon splitting function*, Phys. Lett. B848 (2024) 138351 [arXiv:2310.01245]

[23] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt, *Additional moments and x-space approximations of four-loop splitting functions in QCD*, Phys. Lett. B849 (2024) 138468 [arXiv:2310.05744]

[24] G. Falcioni, F. Herzog, S. Moch, A. Pelloni, and A. Vogt. *Four-loop splitting functions in QCD – The quark-to-gluon case*, Phys. Lett. B856 (2024) 138906 [arXiv:2404.09701]

[25] G. Falcioni, F. Herzog, S. Moch and S. Van Thurenhout, *Constraints for twist-two alien operators in QCD*, JHEP 11 (2024) 080 [arXiv:2409.02870]

[26] T. Gehrmann, A. von Manteuffel and T.Z. Yang, *Leading Twist-Two Gauge-Variant Counterterms*, PoS LL2024 (2024) 087 [arXiv:2409.10303]

[27] G. Falcioni, F. Herzog, S. Moch, A. Pelloni, and A. Vogt. *Four-loop splitting functions in QCD – The gluon-gluon case*, Phys. Lett. B860 (2025) 139194 [arXiv:2410.08089]

PoS(LL2026)022

[28] B.A. Kniehl and V.N. Velizhanin, *Four-loop anomalous dimension of flavor non-singlet twist-two operator of general Lorentz spin in QCD: $\zeta(3)$ term*, Phys. Rev. Lett. 134 (2025) 131901 [arXiv:2503.20422]

[29] B.A. Kniehl, S.O. Moch, V.N. Velizhanin and A. Vogt, *Flavor Non-singlet Splitting Functions at Four Loops in QCD: Fermionic Contributions*, Phys. Rev. Lett. 135 (2025) 071902 [arXiv:2505.09381]

[30] G. Falcioni, F. Herzog, S. Moch, A. Pelloni and A. Vogt, *Additional results on the four-loop flavour-singlet splitting functions in QCD*, Phys. Lett. B875 (2026) 140278 [2512.10783]

[31] B.A. Kniehl, S.O. Moch, V.N. Velizhanin and A. Vogt, *Four-Loop Gluon Anomalous Dimension of General Lorentz Spin: Transcendental Part*, Phys. Rev. Lett. 137 (2026) 081901 [arXiv:2604.25833]

[32] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T. Yang, *The four-loop non-singlet splitting functions in QCD*, arXiv:2604.09534

[33] S. Moch and A. Vogt, *Properties and implications of the four-loop non-singlet splitting functions in QCD*, arXiv:2605.03889

[34] J.A.M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, Int. J. Mod. Phys. A14 (1999) 2037 [hep-ph/9806280]

[35] Yu.L. Dokshitzer, G. Marchesini and G.P. Salam, *Revisiting parton evolution and the large-x limit*, Phys. Lett. B634 (2006) 504 [hep-ph/0511302]

[36] Yu.L. Dokshitzer and G. Marchesini, *$N = 4$ SUSY Yang-Mills: three loops made simple(r)*, Phys. Lett. B646 (2007) 189 [hep-th/0612248]

[37] B. Basso and G.P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order*, Nucl. Phys. B775 (2007) 1 [hep-th/0612247]

[38] M. Strohmaier, *Conformal symmetry breaking and evolution equations in Quantum Chromodynamics*, PhD thesis, Regensburg 2018, doi:10.5283/epub.37432

[39] A. Mitov, S. Moch and A. Vogt, *Next-to-Next-to-Leading Order Evolution of Non-Singlet Fragmentation Functions*, Phys. Lett. B638 (2006) 61 [hep-ph/0604053]

[40] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025

[41] J. Kuipers, T. Ueda, J. Vermaseren and J. Vollinga, *FORM version 4.0*, Comput. Phys. Comm. 184 (2013) 1453 [arXiv:1203.6543]

[42] B. Ruijl, T. Ueda and J. Vermaseren, *FORM version 4.2*, arXiv:1707.06453

[43] A. Vogt et al., *Progress on double-logarithmic large-x and small-x resummations for (semi-) inclusive hard processes*, PoS LL2012 (2012) 004 [arXiv:1212.2932]

[44] J. Davies, C.H. Kom, S. Moch and A. Vogt, *Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering*, JHEP 08 (2022) 135 [arXiv:2202.10362]

[45] B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams*, Comput. Phys. Comm. 253 (2020) 107198 [arXiv:1704.06650]

[46] J.M. Henn, G.P. Korchemsky and B. Mistlberger, *The full four-loop cusp anomalous dimension in $\mathcal{N} = 4$ super Yang-Mills and QCD*, JHEP 04 (2020) 018 [arXiv:1911.10174]

[47] A. von Manteuffel, E. Panzer and R.M. Schabinger, *Cusp and collinear anomalous dimensions in four-loop QCD from form factors*, Phys. Rev. Lett. 124 (2020) 162001 [2002.04617]

PoS(LL2026)022

[48] T. Gehrmann and E. Remiddi, *Numerical evaluation of harmonic polylogarithms*, Comput. Phys. Commun. 141 (2001) 296 [hep-ph/0107173]

[49] E. Remiddi and J. Vermaseren, *Harmonic polylogarithms*, Int. J. Mod. Phys. A15 (2000) 725 [hep-ph/9905237]

[50] T. Gehrmann, *private communication*

[51] J.A. Gracey, *Anomalous dimension of non-singlet Wilson operators at $O(1/n_f)$ in deep inelastic scattering*, Phys. Lett. B322 (1994) 141 [hep-ph/9401214]

[52] T. Becher, P. Hager, S. Jaskiewicz, M. Neubert and D. Schwienbacher, *Factorization Restoration through Glauber Gluons*, Phys. Rev. Lett. 134 (2025) 061901 [arXiv:2408.10308]

[53] T. Becher, P. Hager, S. Jaskiewicz, M. Neubert and D. Schwienbacher, *Low-energy theory of jet processes and PDF factorization*, JHEP 01 (2026) 024 [arXiv:2509.07082]

[54] J. Blümlein and A. Vogt, *On the behavior of non-singlet structure functions at small x*, Phys. Lett. B370 (1996) 149 [hep-ph/9510410]

[55] J. Blümlein, S. Riemersma and A. Vogt, *The Evolution of unpolarized and polarized structure functions at small x*, Nucl. Phys. B Proc. Suppl. 51 (1996) 30 [hep-ph/9608470]

[56] R. Kirschner and L.N. Lipatov, *Double Logarithmic Asymptotics and Regge Singularities of Quark Amplitudes with Flavor Exchange*, Nucl. Phys. B213 (1983) 122

[57] Z. Bajnok, R.A. Janik and T. Lukowski, *Four-loop twist two, BFKL, wrapping and strings*, Nucl. Phys. B816 (2009) 376 [arXiv:0811.4448]

[58] W.L. van Neerven and E.B. Zijlstra, *Order α_s^2 contributions to the deep inelastic Wilson coefficient*, Phys. Lett. B272 (1991) 127

[59] E.B. Zijlstra and W.L. van Neerven, *Contribution of the second order gluonic Wilson coefficient to the deep inelastic structure function*, Phys. Lett. B273 (1991) 476

[60] S. Catani, F. Fiorani, G. Marchesini, *Small-x Behavior of Initial State Radiation in Perturbative QCD*, Nucl. Phys. B336 (1990) 18

[61] S. Catani and F. Hautmann, *High-energy factorization and small-x deep inelastic scattering beyond leading order*, Nucl. Phys. B427 (1994) 475 [hep-ph/9405388]

[62] V.S. Fadin and L.N. Lipatov, *BFKL pomeron in the next-to-leading approximation*, Phys. Lett. B429 (1998) 127, [hep-ph/9802290]

[63] M. Ciafaloni and G. Camici, *Energy scale(s) and next-to-leading BFKL equation*, Phys. Lett. B430 (1998) 349 [hep-ph/9803389]

[64] M. Ciafaloni and D. Colferai, *Dimensional regularisation and factorisation schemes in the BFKL equation at subleading level*, JHEP 09 (2005) 069 [hep-ph/0507106

[65] M. Ciafaloni, D. Colferai, G.P. Salam, A.M. Stasto, *Minimal subtraction vs. physical factorisation schemes in small-x QCD*, Phys. Lett. B635 (2006) 320 [hep-ph/0601200]

[66] G. Soar, S. Moch, J.A.M. Vermaseren and A. Vogt, *On Higgs-exchange DIS, physical evolution kernels and fourth-order splitting functions at large x*, Nucl. Phys. B832 (2010) 152 [arXiv:0912.0369]

[67] A.A. Almasy, G. Soar and A. Vogt, *Generalized double-logarithmic large-x resummation in inclusive deep-inelastic scattering*, JHEP 03 (2011) 030 [arXiv:1012.3352]

[68] A. Vogt, *Resummation of small-x double logarithms in QCD: semi-inclusive electron-positron annihilation*, JHEP 10 (2011) 025 [arXiv:1108.2993]

PoS(LL2026)022