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Jet functions for next-to-leading power factorization

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We discuss the factorization of scattering processes near partonic threshold at next-to-leading power (NLP) in the threshold variable $1-z$, with $z \equiv q^2/\hat{s}$. We review the general structure of power-suppressed contributions both in Soft-Collinear Effective Theory (SCET) and in a direct QCD approach, and discuss the definition of NLP jet functions as gauge-invariant operator matrix elements in QCD. As a controlled check of the resulting factorization formula, we verify it explicitly at one and two loops for the massive electromagnetic form factor in the limit $m^2 \ll s$, using the method of regions. We conclude by outlining the two challenges that remain for a systematic resummation of NLP logarithms: the treatment of endpoint divergences in SCET convolutions, and the extension of jet functions to radiative processes capable of describing an arbitrary number of soft-gluon emissions – the latter being the last missing ingredient for exponentiation, given that the purely soft sector is already understood in terms of generalised webs via the replica trick.

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1. Introduction: particle scattering near threshold

A key challenge in precision phenomenology at hadron colliders is the resummation of large logarithms beyond leading power. As an example, consider Drell-Yan production $q(p_1)\bar{q}(p_2) \to \gamma^*(q) \to l^+l^-$, with $\hat{s} \equiv (p_1 + p_2)^2$ the partonic centre-of-mass energy, and the threshold variable $z \equiv q^2/\hat{s} \to 1$. The partonic cross section admits a singular expansion in $(1-z)$:

$$\Delta_{ab}(z) \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \left[c_n\, \delta(1-z) + \sum_{m=0}^{2n-1} c_{nm} \left[\frac{\ln^m(1-z)}{1-z}\right]_+ + d_{nm} \ln^m(1-z) + \dots \right]. \tag{1}$$

The $\delta(1-z)$ and plus-distribution terms are the leading-power (LP) contributions; the regular logarithms $\ln^m(1-z)$ are suppressed by one power of $(1-z)$ and define the next-to-leading power (NLP) terms.

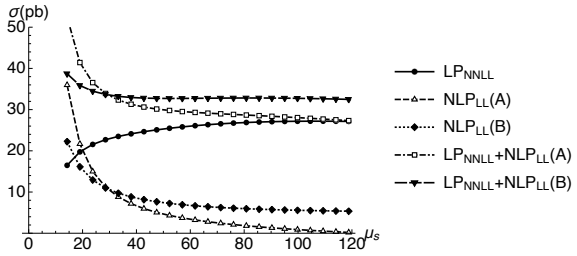

Figure 1: LP vs. NLP resummation in SCET at various logarithmic accuracies for single Higgs production, from [25].

The resummation of LP logarithms is well understood up to N^{3}LL and beyond; see [1–10] for a few seminal works in QCD and SCET. In recent years, a great deal of work has been devoted to understanding the factorization of physical observables beyond leading power, and its implications for the resummation of large logarithms (see e.g. [11–24] for a selection of works, and references therein). Resummation has been achieved in general for leading logarithms (LLs) at NLP. Phenomenological analyses [25, 26] have shown that the LLs at NLP are numerically competitive with NNLLs at LP, see e.g. figures 1 and 2, making their resummation highly relevant for precision physics. Despite substantial progress [27–31], a systematic approach to NLP resummation beyond LL accuracy is still missing.

2. Factorization at leading power

In what follows we continue to consider Drell-Yan near threshold as a case study to discuss the structure of NLP logarithms. The simplicity of LP factorization [1, 2, 7, 8, 10]

$$\frac{d\sigma}{dQ^2} = |C^{A0}|^2 \times f_{a/A} \otimes f_{b/B} \otimes S_{\rm DY}\big[Q(1-z)\big], \tag{2}$$

originates from the fact that large threshold logarithms are generated by the emission of soft gluons only. In the soft limit, a gluon with momentum k^μ emitted from an external line with momentum

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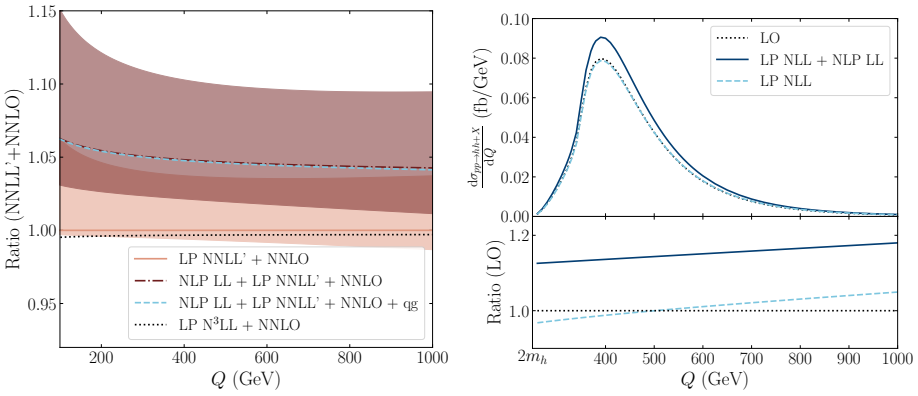

Figure 2: LP vs. NLP resummation in QCD at various logarithmic accuracies for single (left) and double (right) Higgs production, from [26].

p^μ produces an eikonal factor,

$$\mathcal{M}\,\frac{\not{p}-\not{k}}{2p\cdot k}\,\gamma^\mu T^A u(p) \sim \mathcal{M}\,\frac{p^\mu}{p\cdot k}\,T^A u(p), \tag{3}$$

so that multi-gluon emission exponentiates through Wilson lines:

$$\mathcal{M}\,\mathcal{S}\,u(p), \quad \text{with} \quad \mathcal{S} = \langle 0|\Phi_\beta(-\infty,0)|0\rangle, \tag{4}$$

and $\Phi_\beta(\lambda_1,\lambda_2) = \mathcal{P}\exp\left\{ig_s\int_{\lambda_1}^{\lambda_2} d\lambda\,\beta\cdot A(\lambda\beta)\right\}$. The soft function in eq. (2) thus encodes the factorization and exponentiation of soft gluons.

3. Factorization at next-to-leading power

At NLP the factorization structure becomes significantly richer. We distinguish three classes of power-suppressed contributions appearing when factorizing soft and collinear radiation from the hard interaction, either via SCET or via a direct diagrammatic approach in QCD:

- *Next-to-eikonal soft gluons*: emission beyond the eikonal approximation, sensitive, for instance, to the spin of the emitting particle (diagrams (a) and (b) in figure 3), discussed in a direct QCD approach in [32, 33].

- *Direct emission from the hard interaction*: this contribution (diagram (c) in figure 3), first discussed by Low, Burnett and Kroll [34, 35], accounts for the fact that soft emissions begin to resolve the structure of the internal hard non-radiative amplitude at next-to-leading order.

- *Radiative jets*: these describe the emission of soft gluons from a cluster of virtual collinear particles in the same jet sector (diagrams (d), (e) and (f) in figure 3), first identified in [36] and later extended in direct QCD in [11, 37–40]. Within SCET, these structures are identified with collinear matrix elements [17, 20].

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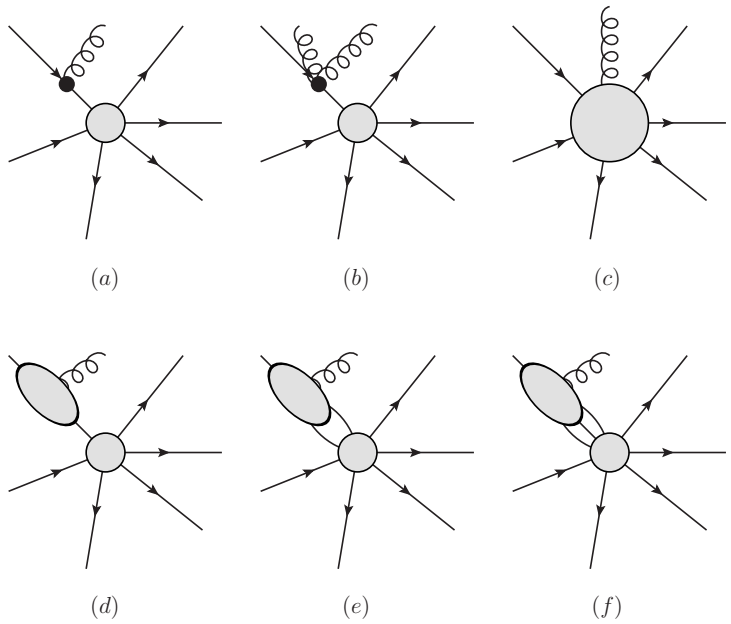

Figure 3: Diagrammatic description of the contributions appearing at NLP.

3.1 NLP factorization in SCET

SCET [41, 42] organizes the Lagrangian as $\mathcal{L}_{\rm SCET} = \sum_i \mathcal{L}_{c_i} + \mathcal{L}_s + \sum_n O_n$, with operators

$$O_n = \int dt_1 \cdots dt_n \, C(t_1, \ldots, t_n) \, \phi_1(t_1 n_{1+}) \cdots \phi_n(t_n n_{n+}). \tag{5}$$

The SCET Lagrangian and operators are constructed so as to reproduce the momentum-region expansion of a scattering cross section. Soft-collinear decoupling at LP guarantees that the LP cross section factorizes as

$$\sigma \sim \mathcal{H} \otimes \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n \otimes \mathcal{S}, \tag{6}$$

where $\mathcal{H}$ is the hard matching coefficient, $\mathcal{J}_i$ are jet functions (matrix elements of collinear fields), and $\mathcal{S}$ is the soft function. Renormalization group equations for each function, which depends on a single scale, then resum large logarithms.

At NLP one must account for subleading-power Lagrangian insertions $\mathcal{L}_{c_i} = \mathcal{L}_{c_i}^{(0)} + \mathcal{L}_{c_i}^{(1)} + \cdots$, where for instance [42]

$$\mathcal{L}_c^{(1)\,\text{gluon}}(x) = \bar{\xi}\left[x_\perp^\mu n_-^\nu W_c g_s F_{\mu\nu}^s(x_-) W_c^\dagger\right] \frac{\bar{n}\!\!\!/_+}{2} \xi, \tag{7}$$

$$\mathcal{L}_c^{(1)\,\text{quark}}(x) = \bar{q}(x_-) W_c^\dagger i \partial\!\!\!/_{\perp c} \, \xi, \tag{8}$$

and power-suppressed operators such as

$$J_\rho^{A0,A1}(t, \bar{t}) = \bar{\chi}_{\bar{c}}(\bar{t} n_-) n_{+\rho} i \partial_\perp \chi_c(t n_+), \tag{9}$$

$$J_\rho^{A0,B1}(t_1, t_2, \bar{t}) = \bar{\chi}_{\bar{c}}(\bar{t} n_-) n_{\pm\rho} \mathcal{A}_{\perp c}(t_2 n_+) \chi_c(t_1 n_+); \tag{10}$$

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see [14, 43, 44] for further details.

A key complication is that factorization does not lead straightforwardly to resummation: NLP convolutions in SCET suffer from *endpoint divergences*. For instance, in the case of Drell-Yan, the factorization theorem involves a convolution between the collinear and the soft function of the form

$$\int_0^\Omega d\omega \underbrace{\left(n_+ p\, \omega\right)^{-\epsilon}}_{\text{collinear piece}} \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega-\omega)^{\epsilon}}}_{\text{soft piece}}, \tag{11}$$

which is divergent in $D = 4$. As a consequence, standard RGE methods must be supplemented by a non-trivial subtraction procedure [21, 30, 45–47].

3.2 NLP factorization in direct QCD

One of the aims of a direct QCD approach is to determine the exponentiation structure by means of a diagrammatic method, thus avoiding (at least in the first instance) the need to resort to renormalization-group evolution and the problem of endpoint divergences.

In direct QCD several steps are analogous to those of the SCET analysis, with the difference that fields are not divided into soft and collinear modes. One starts by setting up a power counting, obtained by decomposing momenta along the directions of the external particles, $v^\mu = v^+ n_i^\mu + v^- \bar{n}_i^\mu + v_{\perp i}^\mu$, and assigning scalings to all momentum components and masses. In general, the power counting coincides with the scaling assigned by means of expansion by regions [48]. For instance, for Drell-Yan near threshold, collinear momenta scale as $k^\mu \sim Q(1, \lambda, \lambda^2)$ with $\lambda \ll 1$, while soft momenta scale as $k^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$. Next, using reduced diagrams (i.e. diagrams in which off-shell legs are contracted to a point, keeping on-shell lines), one determines the superficial degree of divergence as a function of the number of fermion and photon connections between the hard, soft, and collinear subgraphs [38, 39, 49, 50].

QED Vertex	Suppression
$\bar{\psi}^{(c)}\gamma^\mu\psi^{(c)}A_\mu^{(c)}$	λ
$\bar{\psi}^{(c)}\gamma^\mu\psi^{(c)}A_\mu^{(s)}$	1
$\bar{\psi}^{(s)}\gamma^\mu\psi^{(c)}A_\mu^{(c)}$, $\bar{\psi}^{(c)}\gamma^\mu\psi^{(s)}A_\mu^{(c)}$	1
$\bar{\psi}^{(s)}\gamma^\mu\psi^{(s)}A_\mu^{(s)}$	1

	$m = 0$	$m \sim \lambda Q$
Collinear fermion	λ^{-2}	
Soft fermion	λ^{-2}	λ^{-1}
Collinear photon	λ^{-2}	
Soft photon	λ^{-4}	
Collinear loop	λ^4	
Soft loop	λ^8	

Table 1: Left: power counting for QED vertices depending on the soft or collinear nature of the fields (applies to massive and massless fermions alike). Right: power counting for propagators and loop integrals; if no entry appears for $m \sim \lambda Q$, the scaling is identical to the $m = 0$ case.

This determines all diagrammatic structures contributing to a given scattering amplitude up to the desired power in the expansion. For instance, in QED, with momentum and mass scalings as in

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table 1, the complete list of NLP collinear contributions to the amplitude reads [39]

$$\mathcal{M}^{\rm LP} = \left(\prod_{i=1}^{n} J_{(f)}(\hat{p}_i)\right) \otimes H(\hat{p}_1,\ldots,\hat{p}_n)\, S(n_i\cdot n_j), \tag{12}$$

$$\begin{aligned}\mathcal{M}^{\rm NLP}_{\rm coll} = &+\sum_{i=1}^{n}\left(\prod_{j\neq i} J^j_f\right)\Big[J^i_{f\gamma}\otimes H^i_{f\gamma} + J^i_{f\partial\gamma}\otimes H^i_{f\partial\gamma}\Big]\, S \\ &+\sum_{i=1}^{n}\left(\prod_{j\neq i} J^j_f\right)J^i_{f\gamma\gamma}\otimes H^i_{f\gamma\gamma}\, S + \sum_{i=1}^{n}\left(\prod_{j\neq i} J^j_f\right)J^i_{fff}\otimes H^i_{fff}\, S \\ &+\sum_{1\leq i\leq j\leq n}\left(\prod_{k\neq i,j} J^k_f\right)J^i_{f\gamma}J^j_{f\gamma}\otimes H^{ij}_{f\gamma,f\gamma}\, S + O(\lambda^3)\,. \end{aligned} \tag{13}$$

In the first line the momentum dependence is written explicitly, with $\hat{p}_i^\mu = p_i^+ n_i^\mu$ representing the large momentum component of a collinear momentum along direction i. Contributions at NLP involve multiple particles in the same collinear sector, which induces a convolution over the momentum fraction along the common collinear direction (subleading collinear limits). For instance, the first term in $\mathcal{M}^{\rm NLP}_{\rm coll}$ reads more explicitly:

$$\begin{aligned}&\left(\prod_{j\neq i} J^j_{(f)}\right)\Big[J^i_{(f\gamma)}\otimes H^i_{(f\gamma)} + J^i_{(f\partial\gamma)}\otimes H^i_{(f\partial\gamma)}\Big]\, S \equiv S(\hat{p}_i\cdot\hat{p}_j;\epsilon)\left(\prod_{j\neq i} J_{(f)}(p_j;\epsilon)\right) \\ &\qquad\times\int_0^{p_i^+} d\ell_i^+\Big[J^\nu_{(f\gamma)}(p_i-\hat{\ell}_i,\hat{\ell}_i;\epsilon)\, H_{(f\gamma)\nu}(p_1\ldots;p_i-\hat{\ell}_i,\hat{\ell}_i;\ldots p_n;\epsilon) \\ &\qquad\qquad + J^{\nu\rho}_{(f\partial\gamma)}(p_i-\hat{\ell}_i,\hat{\ell}_i;\epsilon)\, H_{(f\partial\gamma)\nu\rho}(p_1\ldots;p_i-\hat{\ell}_i,\hat{\ell}_i;\ldots p_n;\epsilon)\Big]\,. \end{aligned} \tag{14}$$

The next step consists in finding gauge-invariant operator matrix element definitions for the jet and soft functions in QED/QCD, such that the power-suppressed collinear and soft configurations are correctly reproduced [40]. For instance, for the LP jet one finds

$$J_f(p_1,\bar{n}) = \langle p_1|\bar{\psi}(0)\,\Phi_{\bar{n}}(0,\infty)|0\rangle, \tag{15}$$

whilst the NLP "radiative" jet carrying an additional photon with momentum fraction $\bar{n}\cdot\ell$ in QED reads

$$J^\rho_{f\gamma}(p_1,\bar{n},\bar{n}\cdot\ell) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\, e^{-i\ell(\xi\bar{n})}\langle p_1|\Big[\bar{\psi}(0)\Phi_{\bar{n}}(0,\infty)\Big]\Big[\Phi_{\bar{n}}(\infty,\xi\bar{n})\big(iD^\rho\Phi_{\bar{n}}(\xi\bar{n},\infty)\big)\Big]\Big|0\rangle. \tag{16}$$

The jet and soft functions are defined so as to reproduce the collinear and soft region results obtained by means of a momentum-region expansion, respectively. A key difference with respect to the operators introduced in SCET is that in SCET each operator has a definite power counting, and the SCET Lagrangian is constructed such that collinear and soft matrix elements give rise (in particular in the case of SCET-I) exclusively to collinear and soft contributions. The price to pay is relatively involved Feynman rules associated with the subleading-power Lagrangian and operators. By contrast, each of the functions introduced above is itself organised as a series in the power expansion, with its own well-defined leading order (LP for J_f, N$^{1/2}$LP for $J^\rho_{f\gamma}$, NLP for the others), followed by the full tower of subleading-power terms. Furthermore, the jet functions may also contain soft contributions that need to be removed by a subtraction procedure. On the other hand, one of the advantages of the functions in eq. (13) is that they can be calculated directly in QCD.

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4. A case study: the massive quark form factor for $m^2 \ll s$

The massive electromagnetic form factor with $m^2 \ll s$ provides the simplest non-trivial check of the framework, being the first case in which all NLP jet functions in eq. (13) beyond J_f are required [40, 51] to reproduce the vertex function, defined as follows:

$$V^\mu(p_1, p_2) = \bar{v}(p_2)\, \Gamma^\mu(p_1, p_2)\, u(p_1), \tag{17}$$

where

$$\Gamma^\mu = -ie\, e_q \left[F_1(s, m^2)\gamma^\mu + \frac{1}{2m} F_2(s, m^2)\, i\sigma^{\mu\nu} q_\nu \right]. \tag{18}$$

Calculating the form factors in the limit $m^2 \ll s$ by means of the method of regions is already quite useful for singling out factorization properties beyond leading power. Among the various possible regions,

$$\begin{array}{cc} & k^\mu = (k^+, k^-, k_\perp) \\ \text{hard } (h): & k \sim \sqrt{\hat{s}}\,(\lambda^0, \lambda^0, \lambda^0)\,, \\ \text{collinear } (c): & k \sim \sqrt{\hat{s}}\,(\lambda^0, \lambda^2, \lambda^1)\,, \\ \text{anti-collinear } (\bar{c}): & k \sim \sqrt{\hat{s}}\,(\lambda^2, \lambda^0, \lambda^1)\,, \\ \text{ultra-collinear } (uc): & k \sim \sqrt{\hat{s}}\,(\lambda^2, \lambda^4, \lambda^3)\,, \\ \text{ultra-anti-collinear } (\overline{uc}): & k \sim \sqrt{\hat{s}}\,(\lambda^4, \lambda^2, \lambda^3)\,, \\ \text{semi-hard } (sh): & k \sim \sqrt{\hat{s}}\,(\lambda^1, \lambda^1, \lambda^1)\,, \\ \text{soft } (s): & k \sim \sqrt{\hat{s}}\,(\lambda^2, \lambda^2, \lambda^2)\,, \end{array} \tag{19}$$

the semi-hard and soft regions contribute only in the presence of soft radiation, which is not the case for the form factors. At one loop, the hard, collinear, and anti-collinear regions contribute: for instance, for the form factor F_1 one finds

$$\begin{aligned} F_1^{(1l)}\Big|_h =& \left(\frac{\mu^2}{-\hat{s} - i0^+}\right)^\epsilon \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \zeta_2 + \epsilon\left(-16 + \frac{3\zeta_2}{2} + \frac{14\zeta_3}{3} \right) \right. \\ & \left. + \frac{m^2}{\hat{s}} \left[-\frac{2}{\epsilon} - 6 + \epsilon\left(-16 + \zeta_2 \right) \right] + O(\epsilon^2) + O(\lambda^4) \right\}, \\ F_1^{(1l)}\Big|_c =& \left(\frac{\mu^2}{m^2 - i0^+}\right)^\epsilon \left\{ \frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 4 + \frac{\zeta_2}{2} + \epsilon\left(8 + \zeta_2 - \frac{\zeta_3}{3} \right) \right. \\ & \left. + \frac{m^2}{\hat{s}} \left[\frac{1}{\epsilon} + 5 + \epsilon\left(13 + \frac{\zeta_2}{2} \right) \right] + O(\epsilon^2) + O(\lambda^4) \right\}. \end{aligned} \tag{20}$$

The two-loop calculation (see [52, 53] for the original references with the exact result in QCD) is more involved, but also more revealing. Let us begin by noting that all regions can be identified only by carefully choosing the momentum routing: for instance, in the case of the scalar master integral associated with the diagram in figure 4, all regions can be singled out only by considering both loop-momentum parametrizations, as indicated on the right-hand side of the same figure. A crucial observation is that additional collinear regions (ultra-collinear and ultra-anti-collinear) appear at the level of individual diagrams but cancel in the contribution to the form factors. This fact, already observed at LP in [54], confirmed here at NLP, and proven recently to all orders at LP in [58]), provides an important consistency check for the factorization programme: if new regions

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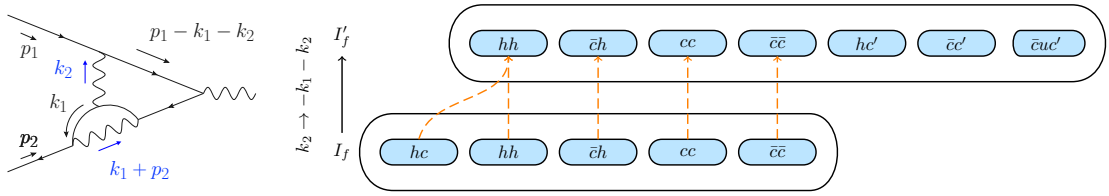

Figure 4: Momentum regions contributing to the scalar integral associated with the diagram on the left-hand side, before and after applying the transformation $k_2 \to -k_1 - k_2$, corresponding to I_f and I'_f respectively. The dashed arrows indicate how the regions in I_f transform accordingly; e.g. the original hc-region maps onto a new hh-region after the collinear mode k_2 mixes with the dominant hard scale associated with k_1. Note how two previously absent regions, $\bar{c}c'$ and $\bar{c}uc'$, and a different hc-region appear after the shift, where the $\bar{c}c'$-region cancels the rapidity divergences present in the cc and $\bar{c}\bar{c}$ regions. Regions that remain invariant are displayed on top of each other, whilst additional regions are shifted outwards, so that all regions are found by collapsing the top row onto its base I_f.

were to contribute at each subsequent loop order, any attempt to establish a factorization theorem would be invalidated.

Once the two-loop form factors have been calculated within the method of regions and checked against the literature, one needs to calculate the jet functions according to their matrix element definitions; see e.g. eqs. (15) and (16). For instance, for J_f and $J_{f\gamma}$ at $O(\alpha_s)$ one finds [40]

$$\begin{aligned} J_f^{(1)}(p_1,\bar{n}) &= i16\pi^2\bar{u}(p_1)\int[dk]\,\frac{\bar{n}\!\!\!/(p\!\!\!/_1 - k\!\!\!/ + m)}{k^2[(p_1-k)^2-m^2][\bar{n}\cdot k]} \\ &= \left(\frac{\bar{\mu}^2}{m^2}\right)^{\epsilon}\bar{u}(p_1)\,\frac{\Gamma(\epsilon)e^{\epsilon\gamma_E}}{\epsilon(1-2\epsilon)}\left(1-\frac{m\epsilon}{p_1^+}\bar{n}\!\!\!/\right), \end{aligned} \tag{21}$$

$$\begin{aligned} J_{f\gamma}^{(1)\rho}(p_1,\bar{n},\ell^+) &= i16\pi^2\,\bar{u}(p_1)\int[dk]\left(\eta^{\rho\sigma}-\frac{\bar{n}^\sigma k^\rho}{\bar{n}\cdot k}\right)\frac{\gamma_\sigma\left(p\!\!\!/_1 - k\!\!\!/ + m\right)}{k^2[(k-p_1)^2-m^2]}\delta(\bar{n}\cdot k-\ell^+) \\ &= \left(\frac{\bar{\mu}^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon)e^{\epsilon\gamma_E}\,\bar{u}(p_1)\,m\left\{x^{1-2\epsilon}\left(-\gamma^\rho+\frac{\bar{n}\!\!\!/\,\hat{p}_1^\rho}{p_1^+}\right)\right. \\ &\quad\left.+\,\frac{m}{p_1^+}\left[\frac{1}{2(1-\epsilon)}\left(\delta(1-x)-(1-2\epsilon)x^{1-2\epsilon}\right)\gamma^\rho\bar{n}\!\!\!/-2x^{-2\epsilon}(1-x)\bar{n}^\rho\right]\right\}. \end{aligned} \tag{22}$$

These results are checked by specialising eq. (13) to the $q\bar{q}$ amplitude in eq. (17) and expanding up to two loops:

$$\begin{aligned} V^{\mu(1)} &= J_f^{(0)}\,H_{f,\bar{f}}^{\mu(1)}\,J_{\bar{f}}^{(0)} + J_f^{(1)}\,H_{f,\bar{f}}^{\mu(0)}\,J_{\bar{f}}^{(0)} + J_f^{(0)}\,H_{f,\bar{f}}^{\mu(0)}\,J_{\bar{f}}^{(1)} \\ &\quad + J_{f\gamma}^{(1)}\otimes H_{f\gamma,\bar{f}}^{\mu(0)}\,J_{\bar{f}}^{(0)} + J_f^{(0)}\,H_{f,\bar{f}\gamma}^{\mu(0)}\otimes J_{\bar{f}\gamma}^{(1)}, \\ V^{\mu(2)} &= J_f^{(0)}\,H_{f,\bar{f}}^{\mu(2)}\,J_{\bar{f}}^{(0)} + J_f^{(2)}\,H_{f,\bar{f}}^{\mu(0)}\,J_{\bar{f}}^{(0)} + J_f^{(0)}\,H_{f,\bar{f}}^{\mu(0)}\,J_{\bar{f}}^{(2)} + J_f^{(1)}\,H_{f,\bar{f}}^{\mu(1)}\,J_{\bar{f}}^{(0)} + \cdots, \end{aligned} \tag{23}$$

where in the two-loop case only the first few terms are displayed. Comparing with the result

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obtained with the method of regions, one has for instance at one loop:

$$V^{\mu(1)}_{f,\bar{f}} = V^{\mu(1)}_{f,\bar{f}}\Big|_h + V^{\mu(1)}_{f,\bar{f}}\Big|_c + V^{\mu(1)}_{f,\bar{f}}\Big|_{\bar{c}}, \tag{24}$$

so that one expects

$$\begin{aligned}
V^{\mu(1)}_{f,\bar{f}}\Big|_h &= J^{(0)}_f \, H^{\mu(1)}_{f,\bar{f}} \, J^{(0)}_{\bar{f}}, \\
V^{\mu(1)}_{f,\bar{f}}\Big|_c &= J^{(1)}_f \, H^{\mu(0)}_{f,\bar{f}} \, J^{(0)}_{\bar{f}} + J^{(1)}_{f\gamma} \otimes H^{\mu(0)}_{f\gamma,\bar{f}} \, J^{(0)}_{\bar{f}}, \\
V^{\mu(1)}_{f,\bar{f}}\Big|_{\bar{c}} &= J^{(0)}_f \, H^{\mu(0)}_{f,\bar{f}} \, J^{(1)}_{\bar{f}} + J^{(0)}_f \, H^{\mu(0)}_{f,\bar{f}\gamma} \otimes J^{(1)}_{\bar{f}\gamma}.
\end{aligned} \tag{25}$$

At two loops one proceeds similarly, and we refer to [40] for further details. We note that the two-loop result involves all types of jet function appearing in eq. (13), so the factorization formula is fully verified by our case study.

5. Conclusions

In these proceedings we have reviewed the current state of the art in the factorization and resummation of hadronic cross sections near threshold at next-to-leading power (NLP). Two complementary frameworks have been discussed: Soft-Collinear Effective Theory (SCET) and a direct diagrammatic approach in QCD.

Within SCET, a factorization theorem at NLP is well established. The cross section factorizes into hard, collinear (jet), and soft functions, each carrying a single characteristic scale. However, proceeding from factorization to resummation meets a fundamental obstacle: the convolutions between collinear and soft functions suffer from endpoint divergences, which prevent a straightforward application of renormalization-group methods. At leading-logarithmic accuracy these divergences can be handled by means of subtraction procedures [21, 30, 45–47], and resummation of LLs at NLP has been achieved in this way for several processes [16, 25, 26]. Beyond LL, however, a systematic treatment of endpoint divergences is still missing, and constitutes the main open problem on the SCET side.

In direct QCD, substantial progress has been made towards a complete factorization theorem at NLP. After setting up a power counting via momentum-region techniques, all diagrammatic configurations contributing at NLP can be classified, and gauge-invariant operator matrix element definitions for the jet and soft functions can be provided. As a non-trivial check of the framework, we have discussed the massive quark form factor in the limit $m^2 \ll s$, for which all NLP jet functions entering the factorization formula (13) are required. The factorization formula has been verified explicitly at one and two loops [40, 51], confirming that only hard and (anti-)collinear regions contribute through NLP, and that ultra-collinear regions present in individual diagrams cancel in the physical amplitude.

To extend the factorization theorem to physical cross sections, one must go beyond the virtual amplitude and include real radiation, thereby making the jet functions *radiative*. The definitions of radiative jet functions used so far in the literature [11, 36] are based on the insertion of a conserved current, which by construction describes the emission of a single soft photon or gluon. For resummation and exponentiation, however, one needs to describe the emission of an arbitrary

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number of soft gluons. Understanding the structure of radiative jets with multiple soft-gluon emissions therefore represents a key open problem for the NLP programme in direct QCD.

Interestingly, the situation is more favourable on the purely soft side. At NLP, both soft gluons and soft quarks can be described within a unified framework based on *generalised soft functions* and *generalised Wilson lines*. The generalised soft function $\mathcal{S}$ can be written as a path integral over generalised Wilson lines, which encode eikonal, next-to-eikonal spin-independent, spin-dependent, and seagull contributions. Applying the replica trick [55], one can show that $\mathcal{S}$ exponentiates in terms of *generalised webs* [33, 56, 57]:

$$\begin{aligned}\widetilde{\mathcal{S}} = \langle 0|F_1 \dots F_n|0\rangle &\sim e^{\sum_{G_e} C_{G_e} \mathcal{W}_{G_e} + \sum_{G_{ne}} C_{G_{ne}} \mathcal{W}_{G_{ne}}} \\ &\sim e^{\sum_{G_e} C_{G_e} \mathcal{W}_{G_e}} \Big(1 + \sum_{G_{ne}} C_{G_{ne}} \mathcal{W}_{G_{ne}}\Big),\end{aligned} \tag{26}$$

where F_n represent generalised Wilson lines, and $\mathcal{W}_{G_e}$, $\mathcal{W}_{G_{ne}}$ are respectively leading- and next-to-leading power (eikonal and next-to-eikonal) webs, provided the phase space for n-gluon emission factorizes into n decoupled single-gluon integrals, a condition satisfied at LP and at LL NLP [57]. The exponentiation of the purely soft sector at NLP is thus, in principle, a solved problem. Given this result, understanding the exponentiation of radiative jets with an arbitrary number of emitted gluons constitutes the last missing ingredient for a complete understanding of the exponentiation structure of QCD cross sections near threshold at NLP.

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