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Large N conformal bootstrap and chirality

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We use the large N critical point formalism to examine the interplay of supersymmetry and chiral symmetry on the location or otherwise of multiple zetas in the renormalization group functions of several related field theories. In particular we determine that the anomalous dimension of the matter field of the Lagrangian corresponding to the bosonic sector of the $O(N)$ Wess-Zumino is free of multiple zetas to $O(1/N^3)$.

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1. Introduction

One topic of interest in quantum field theory over many years has been to understand the structure of perturbative renormalization group functions at very high loop order. In particular cases it has been shown that multiple zeta values will appear in scalar field theories, [1]. More recently Schnetz has demonstrated that new periods will occur at seven loops in scalar ϕ^4 theory [2]. Over the last four decades the general overview is that we now understand the connection of multiple zeta values with loop order in a variety of core theories with different symmetries. Aside from flavour and colour group properties, characterized by the generic parameter N, the main symmetries are supersymmetry, gauge and chiral symmetry. The latter three are strictly only present in integer spacetime dimensions. When perturbative renormalization is carried out in dimensionally regularized extensions one has to address potential inconsistencies that can arise in the regularized version before the limit to the critical dimension can be taken. To tackle such high perturbative orders requires a significant amount of resources. However a particular technique that can probe very high order perturbation theory is the $1/N$ expansion when N is large and it effectively equates to carrying out perturbation theory in a related but different ordering of the underlying Feynman graphs. This method was used to study multiple zetas at high order in powers of $1/N$ but all orders in the coupling constant for scalar field theories in [3] based on the results of [4–6].

In one application of the large N method the $O(N)$ Wess-Zumino model, [7], was examined in [8] to third order. An interesting feature emerged which was that the multiple zetas that arose in the $O(N)$ scalar theories of [3] were *absent* to the same order in $1/N$ in the Wess-Zumino model. That model is a supersymmetric theory of scalar chiral fields which is renormalizable in four dimensions. As it possesses two of the main continuous symmetries of that dimension it was not immediately clear which of the two symmetries was driving the absence of multiple zetas to that order. Indeed it is already well-established that supersymmetry is related to the principle of maximal transcendentality, [9]. In order to test which of the two symmetries is at the heart of the multiple zeta absence one would need to study the theories that relate to the separate boson and fermion sectors of the Wess-Zumino model. While the fermionic sector was examined at $O(1/N^3)$ in [10] the bosonic sector has yet to be accessed to that order in $1/N$. We therefore report on progress with regard to this using the $1/N$ expansion. Here chirality means vertices are composed of directed edges with the flow either completely towards or away from the interaction point.

2. Background

For four dimensional theories where the matter fields reside in a symmetry group such as $O(N)$ or $SU(N)$ the β-function takes the general form

$$\beta(g) = \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} b_{n-1\,r} N^r g^n \tag{1}$$

where $b_{00} = -\epsilon$ in $d = 4 - 2\epsilon$ dimensions and g is a generic coupling constant. For a core set of quantum field theories such as Quantum Chromodynamics (QCD), ϕ^3 and ϕ^4 theory, the Gross-Neveu-Yukawa and Wess-Zumino models they all share the property that

$$b_{11} \neq 0 \;,\; b_{22} = b_{33} = b_{44} = b_{55} = \ldots = 0 \;. \tag{2}$$

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We note that for QCD it is the number of quark flavours, N_f, that these restrictions apply to but not to the number of colours. The coefficients b_{Li} are computed at successive L loop Feynman graphs in perturbation theory. They can also be calculated in an expansion in a different small parameter which is $1/N$ if N is large which corresponds to a reordering of terms

$$\begin{aligned}\beta(g) &= b_{00}g + b_{11}Ng^2 \\ &+ \left(b_{10}g^2 + b_{21}Ng^3 + b_{32}N^2g^4 + b_{43}N^3g^5 + \ldots\right) \\ &+ \left(b_{20}g^3 + b_{31}Ng^4 + b_{42}N^2g^5 + b_{53}N^3g^6 + \ldots\right) + \ldots \end{aligned} \tag{3}$$

When $g = O(\frac{1}{N})$ then successive lines in (3) are $O(1/N)$, $O(1/N^2)$ and $O(1/N^3)$ with respective coefficient sequences $\{b_{00}, b_{11}\}$, $\{b_{p\,p-1}\}$ and $\{b_{q\,q-2}\}$ with $p \geq 1$ and $q \geq 2$.

Alternatively defining

$$\beta(g) = -\epsilon g + (b_{11}N + b_{10})g^2 + Ng^2 \sum_{i=1}^{\infty} \frac{1}{N^i} \beta_i(b_{11}Ng) \tag{4}$$

the functions, β_i that contain the b_{ij}, can be accessed via the critical exponent

$$\omega(\epsilon) = -\frac{1}{2}\beta'(g_c) \equiv \sum_{n=0}^{\infty} \frac{\omega_n(\epsilon)}{N^n} \tag{5}$$

at the Wilson-Fisher fixed point formally giving for example

$$\beta(g) = -\epsilon g + (b_{11}N + b_{10})g^2 - 2b_{11}Ng^2 \int_0^{b_{11}Ng} d\xi\, \frac{\omega_1(\xi)}{\xi^2} + O\left(\frac{1}{N^2}\right) . \tag{6}$$

Similar resummed expressions are straightforward to establish for other renormalization group functions. The coefficients b_{ij} are accessible via a resummation of Feynman graphs where the canonical propagator exponent is replaced by functions of d and N and are determined from the properties of the Wilson-Fisher fixed point of the underlying universal critical theory.

The universal theory is defined by the kinetic terms of the matter field, which lies in an N-tuplet of a flavour symmetry group, and the force-matter interaction. For example the universal theory that contains $O(N)$ ϕ^4 theory the Lagrangian is

$$L = \tfrac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \tfrac{1}{2}\sigma\phi^i\phi^i + f_\sigma(\sigma, \partial\sigma) \tag{7}$$

where the critical coupling is regarded as unity in large N, $1 \leq i \leq N$ and $f_\sigma(\sigma, \partial\sigma)$ represents all σ dependent terms and self-interactions in even critical dimensions. The scaling behaviour of the critical propagators are deduced by ensuring the action remains dimensionless in d dimensions giving

$$\langle \phi^i(x)\phi^j(0)\rangle = \frac{A\delta^{ij}}{(x^2)^\alpha} \;,\; \langle \sigma(x)\sigma(0)\rangle = \frac{B}{(x^2)^\beta} \tag{8}$$

in coordinate space where A and B are x independent amplitudes and the full exponents are, with $d = 2\mu$,

$$\alpha = \mu - 1 + \tfrac{1}{2}\eta \;,\; \beta = 2 - \eta - \chi \,. \tag{9}$$

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Here η is the anomalous dimension of the ϕ field and χ is the anomalous dimension of the cubic vertex. The explicit expressions for the functions such as $\omega_1(\xi)$ are determined in d-dimensions by the large N critical point formalism of [4–6]. Various critical exponents have been computed to all orders in ϵ to $O(1/N^2)$ or $O(1/N^3)$ in [5, 6] and therefore perturbative coefficients are known at high loop order before the corresponding loop computation has been carried out. More importantly *all* orders insight into high order perturbative structures can be deduced.

For example the wave function exponent η in $O(N)$ ϕ^4 theory is known at $O(1/N^3)$ from the large N conformal bootstrap method [6]. It contains a term involving the function $I(\mu)$ defined by a two loop graph

$$I(\mu) \;=\; \frac{\partial}{\partial\Delta} \ln \Pi(\Delta) \Big|_{\Delta \equiv 0} \tag{10}$$

where Δ is the analytic regulator in coordinate space representation and $\Pi(\Delta)$ is defined in Figure 1. The ϵ expansion is known near four dimensions [3, 11] and contains multiple zeta values that agree with the seven loop $O(N)$ ϕ^4 theory computations in [2]. The same integral appears in η at $O(1/N^3)$ in other theories such as Gross-Neveu-Yukawa, [12, 13], and the supersymmetric nonlinear σ model [9]. By contrast for the $O(N)$ Wess-Zumino model $I(\mu)$ is absent, [8]. In that instance the exponent η was found at $O(1/N^3)$ without using the bootstrap formalism as it has not been constructed for superspace. The final expression only involves polygammas [8]. More specifically the functions

$$\begin{aligned}
\hat{\Psi}(\mu) &= \psi(2\mu-3) + \psi(3-\mu) - \psi(\mu-1) - \psi(1) \\
\hat{\Theta}(\mu) &= \psi'(\mu-1) - \psi'(1) \\
\hat{\Phi}(\mu) &= \psi'(2\mu-3) - \psi'(3-\mu) - \psi'(\mu-1) + \psi'(1) \\
\hat{\Omega}(\mu) &= \psi'''(\mu-1) - \psi'''(1)
\end{aligned} \tag{11}$$

appear in the wave function exponent at $O(1/N^3)$.

$\Pi(\Delta) \;=$

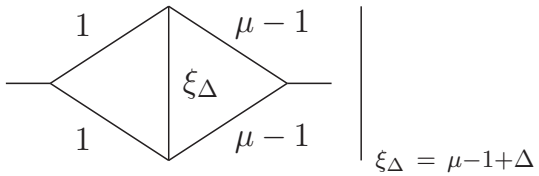

Figure 1: Two loop self-energy graph that defines $I(\mu)$.

The Wess-Zumino model is a cubic theory in chiral superfields Φ and $\bar{\Phi}$ with the superspace interaction action, [7],

$$S_{\rm int} \;=\; \frac{g}{3!} \int_x \left[\int d^2\theta \, \Phi^3(x,\theta,\bar{\theta}) \;+\; \int d^2\bar{\theta} \, \bar{\Phi}^3(x,\theta,\bar{\theta}) \right] \tag{12}$$

where the edges are all directed into or away from a vertex. The component Lagrangian contains scalar Yukawa interactions. A natural question arises as to whether the absence of multiple zeta values is due to supersymmetry or chiral symmetry or both. While the directed vertices means that

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only Feynman graphs with even edge subgraphs contribute to any Green's function, graphs that remain could contain $I(\mu)$ which subsequently cancels in the sum. The way to resolve this is to study the corresponding scalar sector of (12) using the large N conformal bootstrap.

3. Large N scalar chiral theory

The scalar sector of the $O(N)$ Wess-Zumino model involves two doublets of fields $\{\phi^i, \bar{\phi}^i\}$ and $\{\sigma, \bar{\sigma}\}$ where the universal Lagrangian for the large N expansion is

$$L = \partial_\mu \bar{\phi}^i \partial^\mu \phi^i + \sigma \phi^i \phi^i + \bar{\sigma} \bar{\phi}^i \bar{\phi}^i + f_\sigma(\sigma, \bar{\sigma}, \partial\sigma, \partial\bar{\sigma}) \tag{13}$$

with the final term representing the interactions that are active in even critical dimensions. For the large N conformal bootstrap formalism of [6] the coordinate space critical propagators are

$$\langle \phi^i(x) \bar{\phi}^j(0) \rangle = \frac{A \delta^{ij}}{(x^2)^\alpha} \quad , \quad \langle \sigma(x) \bar{\sigma}(0) \rangle = \frac{B}{(x^2)^\beta} \tag{14}$$

where α and β are defined in (9). We have applied the large N bootstrap formalism of [6] to deduce η at $O(1/N^3)$ for (13). In general the method involves analytic regularization with a parameter introduced by $\chi \to \chi + \Delta$; the spacetime dimension $d = 2\mu$ is not the regulator. Briefly the method corresponds to perturbation theory in χ. As the propagators include the anomalous dimensions no edges are decorated. Additionally all vertices are replaced by a conformal triangle, [5, 14, 15], meaning there are no vertex subgraphs. To the requisite order the relevant graphs and bootstrap vertex function is given in Figure 2 where the dot at a vertex indicates a conformal triangle and the double line denotes a σ or $\bar{\sigma}$ field. So only two graphs contribute to the evaluation of η at $O(1/N^3)$ which is denoted by η_3.

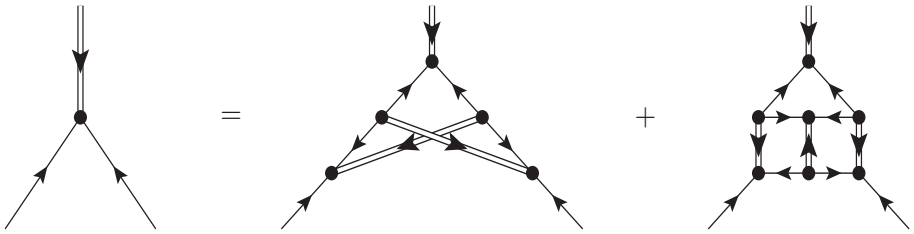

Figure 2: Vertex consistency condition for the large N bootstrap.

Applying the conformal bootstrap formalism of [6] as well as that of [4, 5] we have computed all the core exponents for the fields and corrections to scaling in addition to η_3. In order to verify that the results are credible using Forcer [16–18] we have renormalized the relevant six dimensional Lagrangian to four loops which incorporates (13) and is

$$L = \partial_\mu \bar{\phi}^i \partial^\mu \phi^i + \partial_\mu \bar{\sigma} \partial^\mu \sigma + \frac{g_1}{2} \bar{\sigma} \bar{\phi}^i \bar{\phi}^i + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \bar{\sigma}^3 + \frac{g_2}{6} \sigma^3 \tag{15}$$

to determine the field anomalous dimensions and β-functions as well as the mixing matrix of all of the dimension two operators. Equipped with these the respective critical exponents at the

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Wilson-Fisher fixed point are in full agreement with the ϵ expansion of the $1/N$ exponents when $\mu = 3 - \epsilon$, [19]. Having checked that the perturbative and large N expressions tally we find that in d-dimensions η_3 does not contain either $I(\mu)$ or $\hat{\Omega}(\mu)$. So like the Wess-Zumino model there are no multiple zetas to $O(1/N^3)$ in the ϕ^i field renormalization to all orders in perturbation theory. By way of illustration

$$\eta_3\Big|_{\mu=\frac{3}{2}} = \frac{128[494 - 15\pi^2]}{243\pi^6} \quad , \quad \eta_3\Big|_{\mu=\frac{5}{2}} = -\frac{65536[170503 + 375\pi^2]}{759375\pi^6} \tag{16}$$

in odd dimensions showing that $I(\frac{3}{2}) = 2\ln 2 - \frac{21}{\pi^2}\zeta_3$, [6], is absent.

To complete the investigation into whether multiple zetas are absent in η_3 in the Wess-Zumino model as a result of supersymmetry or due to chirality we recall the situation with the fermion sector. This was examined in [10] where the underlying universal large N Lagrangian was used but based on the chiral Gross-Neveu model of [20]. That Lagrangian is defined in terms of Majorana fermions meaning γ^5 was explicitly present in the interaction. The application of the large N conformal bootstrap computation then followed the parallel path as that of [6]. However since the leading large N term of χ vanished, in order to avoid potential singularities as well as an intermediate $0/0$ issue, a non-abelian generalization of the universal Lagrangian was introduced, [10]. This meant η_3 was computed as a function of Lie group Casimirs. It contained $I(\mu)$ but not $\hat{\Omega}(\mu)$. Moreover the $0/0$ issue could be traced to terms involving the ratio of purely non-abelian Casimirs which were non-problematic due to cancellations prior to taking the abelian limit. Thus in the final expression for η_3 in the chiral Gross-Neveu universality class the coefficient of $I(\mu)$ was zero. The non-abelian generalization could not be readily applied for (13) since the consistency equation would have involved all the topologies of equation (10) of [6].

Having shown that chirality is a driving factor in the absence of multiple zetas appearing at a particular order in large N it is worth discussing other aspects of this symmetry. First one interesting property is that the large N critical exponents of the chiral Gross-Neveu model also precisely reproduce the known perturbative four loop renormalization group functions of the four dimensional chiral XY model, [21]. More specifically we repeated the computation of [10] to three orders in $1/N$ but with the Weyl formulation of the universal theory which is

$$L^{\mathrm{WZf}} = i\bar{\psi}\partial\!\!\!/\psi + \sigma\psi\psi + \bar{\sigma}\bar{\psi}\bar{\psi} + f(\sigma, \bar{\sigma}, \partial\sigma, \partial\bar{\sigma}) \tag{17}$$

and verified that the large N critical exponents were in precise agreement with the $O(\epsilon^4)$ exponents derived from [21]. This observation needs to be viewed in context. Perturbative computations are carried out in the neighbourhood of the Gaussian fixed point. By contrast the large N exponents are constructed at the Wilson-Fisher critical point where the underlying universal theory resides in all spacetime dimensions. It is only when the regularization is lifted in relation to the critical dimension do the Gaussian and Wilson-Fisher fixed points *collide*. In that limit the Lagrangian of the universal theory corresponds precisely to its strictly four dimensional counterpart. Similar features also hold for the large N_f properties of gauge theories and the Wess-Zumino model.

4. Discussion

To summarize we have discussed large N structural properties in different theories in respect of η_3. In particular the large N analysis of the model corresponding to the bosonic sector of the

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Wess-Zumino model has been completed at $O(1/N^3)$. This demonstrated that neither the scalar or fermion sector contain multiple zeta values at this order in the respective matter field wave function renormalization. To answer an earlier question the structural reason for this is the so-called chirality property which excludes graphs which have odd-edge subgraphs rather than supersymmetry. In related fermionic theories, such as the chiral Gross-Neveu model, this translates into the choice of Weyl fermions whence their algebraic properties circumvent the treatment of γ^5 in d-dimensions up to the loop and large N orders studied here. This qualification follows in light of recent four loop computations in [22] which suggest that care has to be taken when treating γ^5 at six loops. If so the large N critical point approach would not be exempt from having to find a parallel resolution which would appear first at $O(1/N^4)$. Another future direction to pursue is to see whether using the universal theory at the Wilson-Fisher fixed point with Weyl fermions for *perturbative* computations could provide an alternative or easier route to treating γ^5 or supersymmetry within dimensional regularization.

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