PROCEEDINGS OF SCIENCE

Holonomic techniques for massive 3-loop form factors: the gluonic case*

J. Blümlein,[a,b] A. De Freitas,[c] P. Marquard,[a] J. Obrovsky[c] and C. Schneider[c,*]

[a] *Deutsches Elektronen-Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, Germany*

[b] *Institut für Theoretische Physik III, IV, TU Dortmund, Otto-Hahn Straße 4, 44227 Dortmund, Germany*

[c] *Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC), Altenberger Straße 69, A-4040, Linz, Austria*

E-mail: Johannes.Bluemlein@desy.de, Peter.Marquard@desy.de, afreitas@risc.jku.at, Jakob.Obrovsky@risc.jku.at, Carsten.Schneider@risc.jku.at

Massive three-loop form factors for vector, axial-vector, scalar, and pseudoscalar currents are vital for precision collider phenomenology. Extending the quarkonic framework to the more complex gluonic case, in Ref. [1] we computed these contributions widely using advanced computer algebra, automated guessing, and large-scale PSLQ searches. Here, we outline the core strategy of the large-moment method and illustrate its main challenges, including solving record-sized recurrences and differential equations. Our high-precision evaluation provides an exact analytic representation around $s = 0$ in terms of MZVs. We achieved a complete analytic series expansion around $s = \pm\infty$ for the first time, which depends on MZVs and three constants out of higher number spaces, along with reliable analytic continuations across $s \in (-\infty, \infty)$, serving as a landmark benchmark for modern symbolic computation.

Loops and Legs in Quantum Field Theory (LL2026)
12-17, April, 2026
Bayreuth, Germany

*DESY-26-108, RISC Report number 26-13

*Speaker

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0) All rights for text and data mining, AI training, and similar technologies for commercial purposes, are reserved. ISSN 1824-8039 . Published by SISSA Medialab.

PoS(LL2026)071

1. Introduction

Massive form factors for vector, axial-vector, scalar, and pseudoscalar currents parameterize the interaction between virtual bosons and massive quark pairs in both Quantum Chromodynamics (QCD) and Quantum Electrodynamics (QED). Acting as indispensable components in heavy-quark precision calculations, these form factors feed into numerous observables vital to collider phenomenology, such as (virtual) Higgs decays and heavy-flavor production. Matching the high accuracy of modern experimental measurements demands matching theoretical precision, driving the evaluation of these quantities to higher loop orders.

Computing three-loop massive form factors represents a particularly challenging problem due to the large number of contributing Feynman diagrams and the complexity of the underlying Feynman integrals, which involve structures beyond standard harmonic polylogarithms. While complete semi–analytic results [2–7], and partial analytic results are available in the literature [8–17], we provided a complete analytic result in the high-energy limit for the quarkonic case in [18] and the gluonic case in [1]. The calculation of the gluonic contributions follows the framework previously established for the quarkonic case. However, because the gluonic case proved to be significantly more complex, its evaluation required new technological advancements and more sophisticated computer algebra techniques. In this note, we summarize the principal steps of our calculation using the method of large moments [19] and illustrate the technical challenges with concrete examples.

2. Calculation steps of the holonomic approach and challenges

The gluonic contributions to the form factors arise exclusively from diagrams in which only gluons are exchanged between the heavy quarks; for typical examples see Fig. 1.

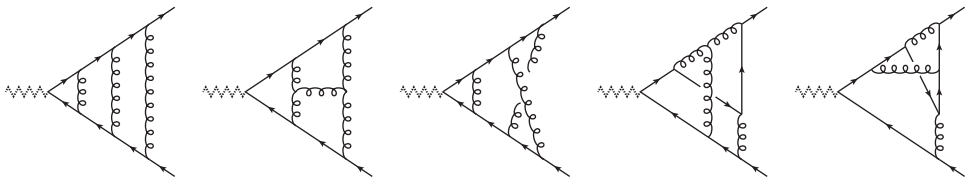

Figure 1: Sample of diagrams required for the calculation of three-loop massive form factors. The solid arrow lines represent massive quarks and the curly lines the gluons, while the dotted zigzag line represents any of the possible external currents.

These diagrams are expressed in terms of dimensionally regularized three-loop Feynman integrals in $D = 4 - 2\varepsilon$ space-time dimensions. The form factors depend on the dimensionless variable s, defined as

$$s = \frac{q^2}{m^2}, \tag{1}$$

PoS(LL2026)071

where q^2 denotes the virtuality of the incoming current and m the heavy-quark mass. Alternatively, we employ the dimensionless variable x, related to s through

$$s = -\frac{(1-x)^2}{x} \implies x = \frac{\sqrt{4-s} - \sqrt{-s}}{\sqrt{4-s} + \sqrt{-s}}; \tag{2}$$

for further details we refer to [1].

Using integration-by-parts (IBP) reduction techniques [20, 21], in particular the software package `Crusher` [22], the form factors F_I are expressed as a linear combination of 417 master integrals M_i (MIs):

$$F_I(s) = c_1(\varepsilon, s)M_1(\varepsilon, s) + \cdots + c_{417}(\varepsilon, s)M_{417}(\varepsilon, s) \tag{3}$$

with rational functions $c_i(\varepsilon, s)$ in s and ε. An important byproduct is that the M_i themselves can be described by a coupled system of first-order linear differential equations. In addition, the initial values for the differential equations are specified at $s = 0$ ($x = 1$), with corresponding expansions given as power series in either s or $y = 1 - x$. In short, the full problem is uniquely determined by (3), the coupled systems in the M_i and their initial values.

Using this input, the overall calculation strategy is based on the large moment method [18] enhanced by advanced analytic continuation. It can be summarized as follows [1].

1 **Generate Master Integral Series:** Compute ρ coefficients in s (or in $1-x$) up to the required order in ε for the power series representations of the master integrals by solving the coupled system of first-order linear differential equations obtained from IBP reductions.

2 **Construct Form Factor Series:** Substitute the truncated power series solutions of the master integrals into Eq. (3) and expand to obtain ρ coefficients in s (or in $1-x$) for the form factors. The resulting coefficients are linear combinations of constants

$$\left\{1,\ \zeta_2,\ \zeta_3,\ \ln(2)\zeta_2,\ \zeta_2^2,\ \mathrm{Li}_4\left(\frac{1}{2}\right),\ \ln^4(2),\ \ln^2(2)\zeta_2,\ \zeta_2\zeta_3,\ \zeta_5\right\} \tag{4}$$

and color factors; $\zeta_z = \zeta(z) = \sum_{i=1}^{\infty} \frac{1}{i^z}$ denotes the Riemann zeta-function and $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$ the classical polylogarithm.

3 **Guess Recurrences and Differential Equations:** For each finite list of rational numbers $(F(n))_{n=0}^{\rho-1}$, guess a linear recurrence relation and a linear differential equation satisfied by the power series $f(s) = \sum_{n=0}^{\infty} F(n)s^n$ (or $f(x) = \sum_{n=0}^{\infty} F(n)(1-x)^n$).

4 **Solve Factorizable Systems:** For components where the recurrence/differential equations factorize into first-order operators, solve the recurrences in terms of iterative sums, and convert its power series representation to iterative integrals.

5 **Analytic Continuation:** For non-first-order factorizable contributions, employ Frobenius' method and extensions, see [23–27] and references therein, to construct high-order local series expansions around regular and singular points ($s \to \pm\infty$, $s = 4$, $s = 16$), matching overlapping expansions to achieve analytic continuation across the full kinematic range $s \in (-\infty, \infty)$.

PoS(LL2026)071

For smaller values ρ, say $\rho = 1000$ it was straightforward to execute Steps 1 and 2 to obtain 1000 coefficients for all of the form factors in x or in s using the techniques described in [18, 28]. In particular, for the contributions up to $O(\varepsilon^{-1})$, we could carry out Step 3 to guess all the underlying recurrences and to solve them in closed form in Step 4. In this way, we rediscovered the results in terms of generalized harmonic sums [29–33] and harmonic polylogarithms [34] that have been calculated earlier in [8–17]. In particular, we succeeded in computing for $O(\varepsilon^0)$ the coefficients of the color factors involving ζ_5 and $\zeta_2\,\zeta_3$. For instance, consider the form factor

$$f(x) = \sum_{k=0}^{\infty} F(n)(1-x)^n \tag{5}$$

for the color factor $C_F^3\zeta_5$ at the projection g_1 of the axial-vector form factor; see [1]. Using the first 30 values of the coefficients

$$\begin{aligned}(F(n))_{n\geq 0} =\Big(&-9760, 0, -\frac{6560}{3}, -\frac{6560}{3}, -2184, -\frac{6544}{3}, -\frac{45848}{21}, -\frac{15328}{7}, -\frac{19796}{9}, -\frac{19904}{9},\\ &-\frac{513928}{231}, -\frac{172376}{77}, -\frac{2899232}{1287}, -\frac{2268976}{1001}, -\frac{20545076}{9009}, -\frac{984128}{429}, -\frac{20785178}{9009},\\ &-\frac{20900336}{9009}, -\frac{616991548}{264537}, -\frac{757990580}{323323}, -\frac{6855857452}{2909907}, -\frac{6888767972}{2909907}, -\frac{159175786100}{66927861},\\ &-\frac{159888055660}{66927861}, -\frac{42257750536}{17612595}, -\frac{62019593516}{25741485}, -\frac{2428552034444}{1003917915}, -\frac{106002999364}{43648605},\\ &-\frac{23657503150756}{9704539845}, -\frac{23744630290016}{9704539845}, \dots\Big)\end{aligned} \tag{6}$$

produced by Steps 1 and 2 we employed the `ore_algebra` package [35, 36] developed in Sage and guessed the recurrence

$$\begin{aligned}&-(n-1)\left(2n^4+n^3+5n^2-60n+48\right)F(n)\\ &\quad+2(4n^5-n^4-n^3-107n^2+69n+156)F(n+1)\\ &\quad+(-10n^5-n^4+34n^3+187n^2+342n-1272)F(n+2)\\ &\quad+2(n+4)(2n^4-7n^3+14n^2-75n+114)F(n+3)=0;\end{aligned} \tag{7}$$

note that the other 970 values (which have not been used for guessing) confirm that the found result is reliable. Next, we used the summation package `Sigma` [37, 38] in the setting of difference rings [39–45] to solve the found recurrence yielding the closed form expression

$$F(n) = -\frac{160\left(13-22n+6n^2+7n^3\right)}{(-1+n)n(1+n)} - 320S_1(n) - 252^{4-n}S_1(2,n) \tag{8}$$

in terms of the harmonic numbers $S_1(n) = \sum_{k=1}^{n}\frac{1}{k}$ and the generalized harmonic sum $S_1(2,n) = \sum_{k=1}^{n}\frac{2^k}{k}$. Finally, using the package `HarmonicSums` [46, 47] and the technologies given in [29–34] we derive the representation

$$f(x) = -\frac{160\left(7+30x+7x^2\right)}{x} - \frac{160\left(2+11x+8x^2+11x^3+2x^4\right)\ln(x)}{(-1+x)x(1+x)}. \tag{9}$$

Note that the number of rational coefficients needed to guess the recurrences depends on the constants (4) under consideration.

PoS(LL2026)071

For the solvable parts, we used the expansion in $1-x$; compare the example above. As it turns out, in order to guess the linear recurrences, one needs fewer coefficients of the expansion in s than in $1-x$. For further considerations we decided therefore to focus on the expansions in s. For the constants ζ_2^2, $\ln^2(2)\zeta_2$, $\ln^4(2)$, $\mathrm{Li}_4\left(\frac{1}{2}\right)$, $\zeta_2\zeta_3$ and ζ_5, 500 rational coefficients suffice. For the case $\ln(2)\zeta_2$ we needed around 6000 coefficients, and for ζ_2 and ζ_3 we required 10000 and 12000 coefficients, respectively. In the most complicated case, namely the constant contribution 1, we could get the recursions using 26000 coefficients. For the underlying techniques to derive so many coefficients in Steps 1 and 2 we refer to [1] and the used ideas given in [18, 28, 48].

In the following we want to give some more details on the most complicated problem: the color factor C_F^3 at the projection g_1 of the axial-vector form factor. Here we needed around 25300 values[1] to guess the linear recurrence. To illustrate the size of these rational numbers in this case, the 26000th value printed in tiny font size requires 18 pages (163879 decimals digits for the numerator and 156054 for the denominator)[2]. For the guessing of the linear recurrence we used an enhanced version of the `ore_algebra` package. The found recurrence (in printed form it needs ~650000 A4-pages) is of the form

$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + a_{91}(n)F(n+91) = 0 \tag{10}$$

where the coefficients $a_i(n)$ are polynomials with integer coefficients with degree up to 4078. For instance, the constant term of the polynomial $a_{91}(n) \in \mathbb{Z}[n]$ is given in Fig. 2 which requires 8207 decimal digits.

Besides the recurrences we also guessed linear differential equations for the form factors

$$f(s) = \sum_{n=0}^{\infty} F(n)s^n \tag{11}$$

from the given coefficients $F(n)$ with $n = 0, \ldots, \rho$. As it turns out, to produce the linear differential equations for $f(s)$ required substantially more coefficients $F(n)$ than guessing the linear recurrences for $F(n)$. For instance, we needed around 37700 values to guess the linear differential equations of order 102 for the color factor C_F^3 at the projection g_1 of the axial-vector form factor. Since we produced 40000 values directly from the large moment method for this particular instance, this calculation could be done without any extra complication. For all the other differential equations we used the available 25000 values (for simpler cases such as ζ_2 and ζ_2 around 10000 or 12000 values) to guess first the linear recurrences and to produce afterwards 40000 values so that we were in the position to guess the differential equations.

Having obtained the governing differential equations, we executed Step 5 to evaluate the expansion coefficients of the form factors across the entire domain $s \in (-\infty, \infty)$. Special attention was given to critical physical points: the high-energy limit $s \to \pm\infty$, the threshold $s = 4$, and the

[1] In a first step we calculated only 25000 values and failed to get the recurrence for this particular instance. By new technologies described in [1] we produced afterwards 40000 values (which was clearly overshooting) to guarantee the final success.

[2] The 40000th coefficient (that we computed from the IBP relations) in printed form requires 40 pages (251779 digits for the numerator and 239755 digits for the denominator.

PoS(LL2026)071

-77472849071423963260505653347182597978538714516798147059382694058279 6 57507529335744200965622299912944778627170731981526760910232692
72320063 0485661823726556356204008752949666798017642632839185287749454251519651 68335787902734937917670509203795548352518251758059981
97160190831642835 5857748115086379919054727150637432800362594260334478395203480919173263 21401463404633945476615106214264468132548253
36124680804998849567073367 6899489160468744411549754738472405332054635355487054057616565930696285 12268798030033899369724377420660726
73074031692970060479125720040241854 8991861745601040637960000135873438491407975567775207268965185724007240 89628917147338814244333624
29354656718937677414503126676340840921521210 7581673780955372163435238535660463727646215990397932252822432712715100 92975737886708121
99977273740311509071909886140074498835819469571569780 2619036803558925507929283307224421642379508274956072525028505674639256 17455395
65934366475507301977072572414715268869577342960801110477247749 3568072229185549197313890493781948335106988508966883431259292198395613
8051188433210497675541583262059638268586211644701118875235220703574962 22690046841926579535936609003757973567059982570006814096085603
87707949 1167112205921599271513253323399945264809210397014195768871502006945386 07912795454210971027624970133174953096386526331297173
95799228644263000 6581792387116046651415609402159176172825775505585244791279825367755512 35448593011553296760901778358635853015933671
01263547462325527405138092 2097409383492441098607864500395622101942569341324077867402527207422609 12327490470299061676146801484588391
22055056519617735258691608162881837 5652460645635611784314030965864425231794516266141691653063689426866027 11745616385696115686174901
69706291855693144618509940153528519993975269 6465383731291360819821573149575544061298319808439909115763103775974165 84945664941426862
44141176281746064301994095097310963520027222967664264 6834134572235007684447754871583492932276891950773179746296400103889155 28889090
63940296046269603942899578229840898704896228238015779262437328 5143297154853772670519152055148935585088925390591315353842998764682732
1026257419451918921689668595582509133337950389991262905953186939760353 64525878339434500356026387099232381494005835336958823372326204
34957613 3273523656700185378833358125423529333192767360954588046186784615293038 43495663879268533720124192394199210594679479005760649
67497257826499040 3472219809366790147463958605751739172139043925538753283600710247821728 82459165885391824511095097410372167474221380
68883719046241026383768318 4041282893107176046729559207452543531790874672565220584132292808858374 86256757455099599326170652555302026
37061159082150066036395045180622179 0943216926967185957328068465887798362784366644770562037091081637414667 11926397019591999903014486
88922622794757755573270151680850553209835330 7875699226272825494893376374255702090977483431204926921810171780275899 52764371636462177
61318220508584853658308433743828702291331751499629379 5728434832552243835359455277720221269605606810258698652243331848410020 96998373
51216775638632358519112366759702331139044817959661161767469801 8135155998167173665354908924919054138338895971892015406108193643150793
5047832005032613678745734636538393816730372700816437716540285600434742 95365774057136657036838773167337409246386417311758120142032614
49477727 8338550436310492487184400863772244333409709540881034962356888905328319 78527583697619509291087943325751664638926653860273707
74096565848565944 6209931572730892744237835035939619265952807421207794826136747213668067 45554290664587449565947623163769405080963888
05517471781817362720928871 7797032325947203654480645284020261261616614593819921088807803577082235 81326780901722219369295580368506520
91173738442643784949787594609335679 3495626848886847816828866704847673217127996376952873036852807133635802 5 00009237028648891798700156
04148768450116020907095368831373629504680649 8071301198150497338222979483171381840488298535276612101026961114847428 33166489110122473
22051400893722929583194231324130581556727213900010055 8758990554450974790910231547331542982980412131411762673217726128618390 20957232
07290500861344845879315119474679825474120823583532123145751521 4655352696304687613335977756494082530712502600559353763729990877213478
7676808642767723952008969807756328666444238643328302225075820969204945 21268165324745799213597541994070874127777819877971984736957828
75749088 0030164248455334905475129437030281929065541362149030870651995588726982 75200436521704504623824191215570734214041097520333691
57195109778116449 6748142546660145771714831524511626474098065456977376200768668078656731 05863762372552529873482023310740701748382821
16537555552494105847656 12 1872451754128157994465672391578154193436267498613747351544623923996857 16755663606061485217113903363990783
85775613545652959625759522990665087 2050834746960880405130032113412131638323524301467297457388133892515772 06070868401760500127458299
40512669351340255776605063923326500235746768 2532374310004222447141820000150569200257907307135870590375790583196546 91280105764420440
21361689686448073379253642027554432811384573029654738 3078418502700774711323764133172536234234574149578644484895450390248006 61161662
89338522721532784374734848631663246408518328431964455453650689 8442519565407567055906470126647994864648150166847161605961719259553561
1285501711446715894282104789037944549954599610359279854765575016442679 21682306136667616697387538852568576591492891228395298682656513
08460350 0816099242103629352076224118342410605210848333990826640871719526074806 74506450860680650683201453077060207847899677449419398
73022138864987330 0402162317957167969085078599018397153640617466254108556042220461484604 23107004263025181151871342894203586049735937
08206467003904135449176531 3359559410227516032325097975955361978124644871950768035111516999857683 79181226045116511028379229243048684
23290529538094251509120612564950403 7134613099178202657501255043104658437610958948395571464625008843020168 05064041693229905936442445
28311965667528452082996385707798419064872492 7923666626946599441064261345436946125975423314228545218713841932341884 96931858668028219
68149546012739890871718413859466353743912692776142637 1243543007714400706075510696116058571352141114572746186416869642091089 44791659
83028996997273582062057447681105276401712344902450651780129987 5917462521662603856939551722852396392723454848357146332104536193978434
1163281455871063299202962505697871614959319733648231024033562141411110 54794614619301209415542977194863765353206951086310200322773539
88651779 2164015159051691558730221379654667964719180249873897742715169946708779 71874352913560050556091116049917648332224762696033131
73554603845284047 9246633066574443144450009376285042939993927099409407588217795529379858 03803433356082328603397800236440745066947511
76994053091637584704752141 9005830805251332053648182982378493594103878685067679569228256895373481 23790465756961788029308687819439446
52118051204476651694583593043756365 9405242541869652504189897107019027155302991938136685327730655458779224 26429697989864094868824529
30893379859750149249748539948066125475457714 9395111929283562102335201295434227863454776031155129017896565062464570 93003923950881890
22789108528268627916541778448810037450839576198124902 0920634303581007557217546386326946635608535129098600580516476995946030 20530197
66298695459320207417234455970779508265867103640609878989367805 7304428708248989988449759554844152647621330904746769195466281518740347
3923604480000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000000000000000000000000000000000000000
00000000 0000000000000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000000000000000000000000000000
00000000000000000 0000000000000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000000000000000000000
00000000000000000000000000 0000000000000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000000000000
00000000000000000000000000000000000 0000000000000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000
00000000000000000000000000000000000000000000 0000000000000000000000000000000000000000000000000000000000000000000000 00000000000000000
00000000000000000000000000000000000000000000000000000 000000000000000000

Figure 2: A typical integer value arising in the linear recurrence of the color form factor $g_1 C_F^3$.

pseudo-threshold $s = 16$. Leveraging the specialized techniques detailed in [1], we successfully determined the expansion coefficients at $s = \pm\infty$ to an accuracy up to 4000 digits. Executing these computations efficiently posed a significant technical challenge due to the immense size of the underlying differential equations. High accuracy demanded dense grids of intermediate points for analytic continuation, as well as careful management of local poles that appear in individual differential equations but systematically cancel out in the final physical form factor. Remarkably, for the expansion around $s = \pm\infty$, this high-precision data allowed us to fit the coefficients via

PoS(LL2026)071

the PSLQ algorithm [49–51] into linear combinations of multiple zeta values (MZVs) and three additional constants beyond the traditional MZV basis:

$$\tilde{\kappa}_1 = \frac{3^5}{2^2}\mathrm{H}_{\{3,0\},0,0}(1) \tag{12}$$

$$\tilde{\kappa}_2 = \frac{3^5}{2^4}\mathrm{H}_{\{3,0\},0,0}(1)\left[\frac{3}{2}\mathrm{H}_{\{3,0\},0}(1) - \mathrm{H}_{\{6,0\},0}(1)\right] \tag{13}$$

$$\tilde{\kappa}_3 = \frac{3^5}{2^2}\mathrm{H}_{\{3,0\},0,0}(1)\mathrm{H}_{\{\{4,-2,1\},0\},0,0}(1) \tag{14}$$

The first two constants are cyclotomic ones [32] and the third belongs to the class of quadratic form induced polylogarithms [52]. We refer the reader to [1] for a comprehensive discussion on these constants and their symbolic derivation.

We remark that all these heavy calculations arising in Steps 1–5 have been carried by massive parallel processing both at the supercomputer MACH-2 (with 1728 processor cores and 20 TB of global shared memory) at JKU and the fast servers of DESY.

3. Conclusion

In Ref. [1], we computed the gluonic contributions to three-loop heavy-quark form factors for vector, axial-vector, scalar, and pseudoscalar currents using advanced computer algebra techniques and high-performance automated guessing algorithms. In this note, we have summarized the core strategy of the large-moment method and highlighted its main computational challenges through concrete examples – most notably involving one of the largest recurrences (and differential equations) ever guessed in contemporary particle physics problems. These high-precision evaluations deliver a fully analytic representation in the low-energy expansion around $s = 0$ and the high energy expansion $s \to \pm\infty$ in terms of multiple zeta values (MZVs) and additional special constants, while enabling reliable numerical analytic continuations across the complete physical region $s \in (-\infty, \infty)$. Beyond their immediate application to precision QCD phenomenology, these results provide a crucial benchmark that demonstrates the power of combining cutting-edge differential equations, large-scale PSLQ integer-relation searches, and state-of-the-art symbolic summation methods.

Acknowledgment. We thank DI J. Messner from Zentraler Informationsdienst (ZID) Johannes Kepler University, Linz, for installing the parallelized guessing software [36] at the supercomputer `MACH-2` of JKU Linz and his help in maintaining the runs determining the largest differential and difference operators in this project. We thank M. Kauers and K. Schönwald for discussions, and D. Broadhurst for his interest in our work. This research was funded in whole or in part by the Austrian Science Fund (FWF) grants DOI 10.55776/P33530, DOI 10.55776/PAT1332123, and DOI 10.55776/I6130.

References

[1] J. Blümlein, A. De Freitas, P. Marquard, J. Obrovsky, C. Schneider, *Analytic results on the massive three-loop form factors: gluonic contributions*. Technical report no. 26-12 in RISC Report Series, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Austria. 2026.

PoS(LL2026)071

[2] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *Singlet and nonsinglet three-loop massive form factors*, Phys. Rev. D **106** (2022) 034029 [arXiv:2207.00027 [hep-ph]].

[3] M. Egner, M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *Three-loop nonsinglet matching coefficients for heavy quark currents*, Phys. Rev. D **105** (2022) no.11, 114007 [arXiv:2203.11231 [hep-ph]].

[4] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *Massive Vector Form Factors to Three Loops*, Phys. Rev. Lett. **128** (2022) 172003 [arXiv:2202.05276 [hep-ph]].

[5] K. Schönwald, *Massive form factors at $O(\alpha_s^3)$*, PoS (LL2022) 006 [arXiv:2207.06705 [hep-ph]].

[6] M. Fael, F. Lange, K. Schönwald and M. Steinhauser, *Massive three-loop form factors: Anomaly contribution*, Phys. Rev. D **107** (2023) no.9, 094017 [arXiv:2302.00693 [hep-ph]].

[7] K. Schönwald, *Singlet and anomaly contributions to massive QCD form factors* PoS (RADCOR2023) 045.

[8] J. Gluza, A. Mitov, S. Moch, and T. Riemann, *The QCD form factor of heavy quarks at NNLO*, JHEP **07** (2009) 001 [arXiv:0905.1137 [hep-ph]].

[9] J.M. Henn, A.V. Smirnov, and V.A. Smirnov, *Analytic results for planar three-loop integrals for massive form factors, JHEP* **12** (2016) 144 [arXiv:1611.06523 [hep-ph]].

[10] J. Henn, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Massive three-loop form factor in the planar limit*, JHEP **1701** (2017) 074 [arXiv:1611.07535 [hep-ph]].

[11] T. Ahmed, J.M. Henn and M. Steinhauser, *High energy behaviour of form factors*, JHEP **06** (2017) 125 [arXiv:1704.07846 [hep-ph]].

[12] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, *Heavy Quark Form Factors at Three Loops in the Planar Limit* Phys. Lett. B **782** (2018) 528–532 [arXiv:1804.07313 [hep-ph]].

[13] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, *Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable*, Nucl. Phys. B **939** (2019) 253–291. [arXiv:1810.12261 [hep-ph]].

[14] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Three-loop massive form factors: complete light-fermion corrections for the vector current*, JHEP **03** (2018) 136 [arXiv:1801.08151 [hep-ph]].

[15] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Three-loop massive form factors: complete light-fermion and large-N_c corrections for vector, axial-vector, scalar and pseudo-scalar currents*, JHEP **05** (2018) 187 [arXiv:1804.07310 [hep-ph]].

[16] J. Blümlein, P. Marquard and N. Rana, *Asymptotic behavior of the heavy quark form factors at higher order* Phys. Rev. D **99** (2019) no.1, 016013 [arXiv:1810.08943 [hep-ph]].

PoS(LL2026)071

[17] J. Blümlein, P. Marquard, N. Rana and C. Schneider, *The Heavy Fermion Contributions to the Massive Three Loop Form Factors*, Nucl. Phys. B **949** (2019) 114751 [arXiv:1908.00357 [hep-ph]].

[18] J. Blümlein, A. De Freitas, P. Marquard, N. Rana and C. Schneider, *Analytic results on the massive three-loop form factors: Quarkonic contributions*, Phys. Rev. D **108** (2023) no.9, 094003 [arXiv:2307.02983 [hep-ph]].

[19] J. Blümlein and C. Schneider, *The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory*, Phys. Lett. B **771** (2017) 31–36, [arXiv:1701.04614 [hep-ph]].

[20] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β Functions in 4 Loops*, Nucl. Phys. B **192** (1981) 159–204.

[21] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, Int. J. Mod. Phys. A **15** (2000) 5087–5159 [hep-ph/0102033].

[22] P. Marquard and D. Seidel, *The* `Crusher` *algorithm*, unpublished.

[23] F.G. Frobenius. *Ueber die Integration der linearen Differentialgleichungen durch Reihen*, J. reine angew. Math. (Crelle) **76** (1873) 214–235.

[24] R. Courant and D. Hilbert, Methods of mathematical physics, I and II, (Springer, Berlin 1024; Interscience Publishers, New York, 1853, 1962).

[25] J. van der Hoeven. *Fast evaluation of holonomic functions near and in singularities*. J. Symbol. Comput. **31** (2001) 717–743.

[26] M. Mezzarobba, *Rigorous multiple-precision evaluation of D-finite functions in Sagemath*, 5th International Congress on Mathematical Software (ICMS 2016), Berlin, Germany (2016), arXiv:1607.01967v1 [cs.SC].

[27] S.A. Abramov, D. Khmelnov, A. Ryabenko, *On truncated series involved in exponential-logarithmic solutions of truncated LODEs*, In: Computer Algebra in Scientific Computing, CASC 2022, pp. 18–28, 2022.

[28] J. Blümlein, P. Marquard and C. Schneider, *A refined machinery to calculate large moments from coupled systems of linear differential equations*, POS (RADCOR2019) 078, pp. 1–15 [arXiv:1912.04390 [cs.SC]].

[29] J.A.M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, Int. J. Mod. Phys. A **14** (1999) 2037–2076 [hep-ph/9806280].

[30] J. Blümlein and S. Kurth, *Harmonic sums and Mellin transforms up to two loop order*, Phys. Rev. D **60** (1999) 014018 [hep-ph/9810241].

[31] S. Moch, P. Uwer and S. Weinzierl, *Nested sums, expansion of transcendental functions and multiscale multiloop integrals*, J. Math. Phys. **43** (2002) 3363–3386 [hep-ph/0110083].

PoS(LL2026)071

[32] J. Ablinger, J. Blümlein and C. Schneider, *Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials*, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063 [math-ph]].

[33] J. Ablinger, J. Blümlein and C. Schneider, *Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms*, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378 [math-ph]].

[34] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, Int. J. Mod. Phys. A **15** (2000) 725–754 [hep-ph/9905237].

[35] M. Kauers, *Guessing Handbook*, Technical report no. 09–07 in RISC Report Series, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Austria. 2009.

[36] M. Kauers, M. Jaroschek, and F. Johansson, *Ore Polynomials in Sage*, in: Computer Algebra and Polynomials, Editors: J. Gutierrez, J. Schicho, Josef, M. Weimann, Eds.. Lecture Notes in Computer Science **8942** (Springer, Berlin, 2015) 105–125 [arXiv:1306.4263 [cs.SC]].

[37] C. Schneider, *Symbolic Summation Assists Combinatorics*, Sém. Lothar. Combin. **56** (2007) 1–36 article B56b.

[38] C. Schneider, *Simplifying Multiple Sums in Difference Fields*, in: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions Texts and Monographs in Symbolic Computation eds. C. Schneider and J. Blümlein (Springer, Wien, 2013) 325–360 [arXiv:1304.4134 [cs.SC]].

[39] M. Karr, *Summation in finite terms*, J. ACM **28** (1981) 305–350.

[40] M. Bronstein, *On solutions of linear ordinary difference equations in their coefficient field*, J. Symbolic Comput. **29** (2000), pp. 841–877.

[41] S.A. Abramov, M. Petkovšek, *D'Alembertian solutions of linear differential and difference equations* in Proc. ISSAC'94, ed von zur Gathen J. (ACM Press) pp. 169–174.

[42] C. Schneider, *A refined difference field theory for symbolic summation*, J. Symbolic Comput. **43** (2008) 611–644 [arXiv:0808.2543].

[43] C. Schneider, *A difference ring theory for symbolic summation*, J. Symb. Comput. **72** (2016) 82–127 [arXiv:1408.2776 [cs.SC]].

[44] C. Schneider, *Refined telescoping algorithms in $R\Pi\Sigma$-extensions to reduce the degrees of the denominators*, In: Proc. ISSAC '23, pp. 498-507. arXiv:2302.03563 [cs.SC].

[45] C. Schneider, S.A. Abramov, M. Bronstein, M. Petkovšek, C. Schneider, *On rational and hypergeometric solutions of linear ordinary difference equations in $\Pi\Sigma^*$-field extensions*, J. Symbolic Comput. **107** (2021) 23–66 [arXiv:2005.04944].

[46] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, Diploma Thesis, JKU Linz, 2009, arXiv:1011.1176 [math-ph].

PoS(LL2026)071

[47] J. Ablinger, *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, Linz U. (2012) arXiv:1305.0687 [math-ph].

[48] J. Blümlein, A. De Freitas, P. Marquard, and C. Schneider. *Challenges for analytic calculations of the massive three-loop form factors.* In P. Marquard and M. Steinhauser, editors, Proceedings of Loops and Legs in Quantum Field Theory, volume PoS(LL2024)031, pages 1–18, 2024. arXiv:2408.07046 [hep-ph].

[49] H. Ferguson and R. Forcade, *Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two*, Bull. Am. Math. Soc. **1** (1979) 912–914.

[50] H. Ferguson and D. Bailey, *A Polynomial Time, Numerically Stable Integer Relation Algorithm*, Technical Report RNR-91-032 (1991).

[51] D.H. Bailey and D.J. Broadhurst, *Parallel integer relation detection: Techniques and applications*, Math. Comput. **70** (2001) 1719–1736 [arXiv:math/9905048 [math.NA]].

[52] J. Ablinger, J. Blümlein and C. Schneider, *Iterated integrals over letters induced by quadratic forms*, Phys. Rev. D **103** (2021) no.9, 096025 [arXiv:2103.08330 [hep-th]].

PoS(LL2026)071